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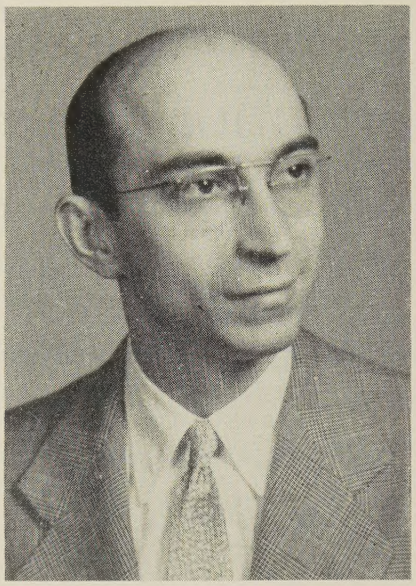

Lotfi A. Zadeh

Lotfi A. Zadeh was born on February 4, 1921, in Baku, Russia. He attended the American College in Teheran, Iran, and was awarded the B.S. degree in electrical engineering by the University of Teheran in 1942. He came to the United States in 1944 and entered Massachusetts Institute of Technology where he received the M.S. degree in 1946. Then he joined the staff of Columbia University as an instructor in electrical engineering and received the Ph.D. degree from Columbia in 1949. He was promoted to assistant professorship in 1950, associate professorship in 1953, and full professorship in 1957. In 1956, he was a member of the Institute for Advanced Study, Princeton, N. J.

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What Is Optimal?

LOTFI A. ZADEH

How reasonable is our insistence on optimal solutions? Not too long ago we were content with designing systems which merely met given specifications. It was primarily Wiener's work on optimal filtering and prediction that changed profoundly this attitude toward the design of systems and their components. Today we tend, perhaps, to make a fetish of optimality. If a system is not "best" in one sense or another, we do not feel satisfied. Indeed, we are apt to place too much confidence in a system that is, in effect, optimal by definition.

To find an optimal system we first choose a criterion of performance. Then we specify a class of acceptable systems in terms of various constraints on the design, cost, etc. Finally, we determine a system within the specified class which is "best" in terms of the criterion adopted. Is this procedure more rational than the relatively unsophisticated approach of the pre-Wiener era?

I am not sure it is. It seems to me that we have merely traded one kind of arbitrariness for another. To begin with, when we choose a criterion of performance, we generally disregard a number of important factors. Moreover, we oversimplify the problem by employing a scalar loss function. Vector-valued loss functions of the type considered by Lionel Weiss may be more appropriate in some cases, but their use presents many technical difficulties. In a few instances, such as the final-value problem considered by Booton, one can show that an optimal solution in terms of some simple criterion such as the mean-square error, also is optimal for a broad class of criteria. Although this is comforting, it does not resolve the arbitrariness inherent in the choice of a criterion of performance.

Another and perhaps more serious difficulty concerns the rational choice of decision functions under uncertainty. What should be done when the probabilities of the "states of nature" characterizing a problem are not known? Here we have several basic principles to choose from, each having its partisans and opponents. The oldest principle, "the principle of insufficient reason," attributed variously to Laplace or Bernoulli, states that if the *a priori* probabilities of states are not known, they should be assumed to be equal. A serious and long-recognized weakness of this rule is that the states can be defined in many different ways, and a uniform distribution for some particular identification of states might be nonuniform for other identifications. Nevertheless, this principle has its proponents, and, in fact, in a recently suggested axiomatic system due to Herman Chernoff it appears as a theorem.

If one could assume that designing a system subject to random inputs is a two-person zero-sum game against nature, then the minimax principle of Wald would be rational. Unfortunately, nature can hardly be regarded as an opponent whose loss is the designer's gain. In a modification of the minimax principle which was suggested by Savage, the "payoff" to nature is the "regret," that is, the difference between the maximum payoff which could be obtained if the state of nature were known, and the risk incurred by selecting a particular design in the absence of such knowledge. This principle, too, has its drawbacks, and, like the minimax principle, is excessively pessimistic.

The optimism-pessimism index criterion of Hurwitz attempts to resolve this difficulty by taking a position somewhere between extreme pessimism and extreme optimism. Unfortunately, under this rule it is not true that a randomization over optimal designs remains optimal under the same rule—contrary to what we expect of any rational criterion.

Still another possibility is to consider a set of criteria and select a design which is optimal under a majority of these. The chief objection to this approach is that the ordering of designs using such a procedure may be intransitive. For example, based on the minimax criterion, a design D_1 might be better than D_2 which, in turn, is better than D_3. On the other hand, under Laplace's and Savage's criteria the ordering might be D_2 better than D_3 better than D_1, and D_3 better than D_1 better than D_2, respectively. Thus, a majority of the criteria rank D_3 over D_1, D_1 over D_2, and D_2 over D_3, which is obviously inconsistent.

At present no completely satisfactory rule for selecting decision functions is available, and it is not very likely that one will be found in the foreseeable future. Perhaps all that we can reasonably expect is a rule which, in a somewhat equivocal manner, would delimit a set of "good" designs for a system. In any case, neither Wiener's theory nor the more sophisticated approaches of decision theory have resolved the basic problem of how to find a "best" or even a "good" system under uncertainty.

A Systematic Approach to a Class of Problems in the Theory of Noise and Other Random Phenomena —Part III, Examples*

A. J. F. SIEGERT†

Summary—The method of Part I is applied to the problem of finding the characteristic function for the probability distribution of $\int_0^t \sum_{jk} x_j(\tau)\ K_{jl}(\tau)x_l(\tau)\ d\tau$, where $x_j(\tau)$ denotes the jth component of a stationary n-dimensional Markoffian Gaussian process. The problem is reduced to the problem of solving $2n$ first-order linear differential equations with initial conditions only. For the case of constant K, the explicit solution is given in terms of the eigenvalues and the first $2n - 1$ powers of a constant $2n \times 2n$ matrix. For the case of a symmetric correlation matrix which commutes with K, the problem is reduced to the one-dimensional case treated in Part II. For the case $K_{ij}(t) = \delta_{i1}\delta_{j1}e^{-t}$, where the functional represents the output of a receiver consisting of a lumped circuit amplifier, a quadratic detector, and a single-stage amplifier, the solution has been obtained in a form which is more explicit than that provided by the earlier methods.

I. Introduction

In this part, the method described in Part I[1] is applied to the problem of finding the characteristic function for the probability distribution of $\mathfrak{F} \equiv \int_0^t \sum_{i,k} x_i(\tau)K_{jl}(\tau)x_l(\tau)d\tau$, with or without conditions on $x_i(0)$ and $x_i(t)$, where the functions $x_i(\tau)$ are the components of a stationary n-dimensional Markoffian Gaussian process $x(\tau)$.

The older method of attacking this reduces it to the problem of finding the solution of a certain integral equation. Special cases, for instance the probability distribution of the output of a radio receiver with square law detector and the probability distribution of the filtered output of a multiplier,[2] have been treated in this way. Another special case of interest is the joint distribution of the sample cross-correlation coefficients.

In its original form the older method required finding the eigenvalues of a homogeneous integral equation and the calculation of the Fredholm determinant as an infinite product. It is possible to avoid the latter part of the calculation by solving instead an inhomogeneous integral equation, which is essentially the equation for the Volterra reciprocal kernel; for the special case of the Markoff process, the integral equation can be reduced to a differential equation.[3] However, the actual determination of that solution of the differential equation which also solves the integral equation requires a rather tedious procedure of satisfying boundary and matching conditions even if the differential equation can be solved. It is not surprising, therefore, that the number and type of cases which have been solved explicitly have been very restricted.

While there are approximation methods valid in limiting cases, the exploration for cases which are exactly soluble seems to be of importance, since in Part I a perturbation formalism was derived, which permits the calculation of the characteristic function for functionals "in the neighborhood" of one for which the solution can be obtained. For this purpose, one needs not only the characteristic function of $\mathfrak{F}$, but also a function $\hat{r}$ which is similar to a joint characteristic function and defined as

$$\hat{r}(\eta_1, \eta_2 \cdots \eta_n; \xi_1, \xi_2 \cdots \xi_n; t, \lambda) = \left\langle \exp\left\{i \sum_{k=1}^{n} (\eta_k x_k(0) + \xi_k x_k(t)) - \lambda\mathfrak{F}\right\}\right\rangle_{\text{Av}}. \qquad (1)$$

In Part II we studied this function for the Uhlenbeck process $x(t)$ and $\mathfrak{F} = \int_0^t K(\tau)\ x^2(\tau)d\tau$ and found that it is of the form

$$\hat{r}(\eta, \xi; t, \lambda) = f\cdot\exp\{-\tfrac{1}{2}(a\eta^2 + 2b\xi\eta + c\xi^2)\}, \qquad (2)$$

where f, a, b, and c are functions of t only, and satisfy a system of first-order differential equations with initial conditions only. The function c specially was found to be the solution of a Riccati equation, which could be converted by standard procedure into a linear second-order differential equation for a function u, again with initial conditions only. If u could be determined, then f, a, and b could be expressed in terms of u, by quadratures or more simply; we found, for instance, that f was given simply by

$$f(t) = \{u(0)/u(t)\}^{1/2}.$$

Since f was expressed by the older method as the reciprocal square root of the Fredholm determinant of the kernel $e^{-\beta|\tau_1-\tau_2|}\ K(\tau_2)$, this showed that the Fredholm determinant of this particular kernel satisfies a second-order linear differential equation as a function of the upper limit of the integral equation. Furthermore, the functions

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[1] D. A. Darling and A. J. F. Siegert, "A Systematic Approach to a Class of Problems in the Theory of Noise and Other Random Phenomena—Part I," The RAND Corp., Santa Monica, Calif., Paper P-738; September 10, 1955, and IRE Trans. on Information Theory, vol. 3, pp. 32–37; March, 1957. See also, A. J. F. Siegert, "Part II, Examples," The RAND Corp., Paper P-730; September, 1955, and IRE Trans. on Information Theory, vol. 3, pp. 38–43; March, 1957.

[2] D. G. Lampard, "The probability distribution for the filtered output of a multiplier whose inputs are correlated, stationary, Gaussian time-series," IRE Trans. on Information Theory, vol. 2, pp. 4–11; March, 1956. References to earlier publications using the older method are given in this paper.

[3] A. J. F. Siegert, "Passage of stationary processes through linear and non-linear devices," IRE Trans. on Information Theory, vol. 3, pp. 4–25; March, 1954.

, b, and c are essentially the values $g(0, 0)$, $g(0, t)$, and (t, t) of the Volterra reciprocal kernel $g(\tau_1, \tau_2)$; one hus had the result that these special values of g can be btained trivially from the Fredholm determinant, for he special form of the kernel,[4] so that it is not necessary o solve the integral equation in order to obtain these pecial values of g.

In the present paper these results have been generalized o the case of quadratic functionals of a stationary, -dimensional Markoffian Gaussian random process. The nalogs of the functions a, b, and c are now matrices of n ows and columns, which are functions of t. In Section II ve obtain two first-order linear differential equations, vith initial conditions only, for the matrices $u \equiv b^{-1}$ and $\equiv cb^{-1}$. The Fredholm determinant of the older method ow reduces, but for a trivial factor, to the determinant f u. In Section III an explicit solution is given for the ase of constant K, which requires only the finding of he eigenvalues and the first $2n-1$ powers of a ertain $2n \times 2n$ matrix with constant elements, or lternatively the eigenvalues and eigenvectors of this natrix. The case where the correlation matrix is symnetric and commutes with K has been reduced in ection IV to the one-dimensional problem. In Section V he principal equation of the present method was derived irectly by means of the older method. The characteristic unction for the probability distribution of $\int_0^\infty e^{-t}x_1^2(t)dt$ as been obtained in Section VI in an explicit form, as he reciprocal square root of an $n \times n$ determinant whose lements are series of hypergeometric type.

The linear first-order equations, of course, are in genral not easy to solve. They seem to be, however, a more conomical formulation of the problem, freed of the ecessity of solving the integral equation, in a case where ne is interested only in the Fredholm determinant and ertain special values of the reciprocal kernel. Therefore, hey also may provide a more convenient basis for umerical computation.

II. Reduction of the Problem to Linear First-Order Differential Equations

The n-dimensional stationary Gaussian process $X(t) = \{x_1(t), x_2(t), \cdots x_n(t)\}$ with mean zero is described by he correlation matrix $R(\tau)$ with components[5]

$$R_{kl}(\tau) = \langle x_k(t)x_l(t+\tau)\rangle_{\text{Av}}; \qquad \tau \geq 0 \tag{3}$$

$$R(\tau) = \tilde{R}(-\tau) \tag{4}$$

here the tilde denotes transposed matrix. For stationary Markoffian Gaussian process, $R(\tau)$ has the form

$$R(\tau) = e^{Q\tau} \tag{5}$$

for $\tau \geq 0$, if $R(0)$ is the unit matrix.[6]

The characteristic function χ_2 of the joint probability distribution of $X(0)$ and $X(\tau)$ is then

$$\begin{aligned}\chi_2(\eta_1, &\cdots \eta_n, \xi_1, \cdots \xi_n, t)\\ &\equiv \left\langle \exp\left[i \sum_k (\eta_k x_k(0) + \xi_k x_k(t))\right]\right\rangle_{\text{Av}}\\ &= \exp\left\{-\frac{1}{2}\left[\sum_k (\eta_k^2 + \xi_k^2) + 2\sum_{k,l} \eta_k R_{kl}(t)\xi_l\right]\right\}. \end{aligned}\tag{6}$$

The characteristic function χ_2 satisfies the differential equation

$$\frac{\partial \chi_2}{\partial t} = \sum_{j,l}\left(\frac{\partial \chi_2}{\partial \xi_j} + \xi_j\chi_2\right)Q_{jl}\xi_l \tag{7}$$

which can be verified directly using (5).

To obtain the characteristic function for the distribution of the functional

$$\mathfrak{F} \equiv \int_0^t \sum_{j,l} x_j(\tau)K_{jl}(\tau)x_l(\tau)\, d\tau \tag{8}$$

[with or without conditions on $X(0)$ and $X(t)$] where K is a symmetric matrix, we define the characteristic function for the joint distribution of $X(0)$, $X(t)$, and u,

$$\begin{aligned}\hat{r}(\eta_1 \cdots \eta_n; \xi_1, \cdots \xi_n, t, \lambda) \equiv \langle \exp\{i &\sum_k (\eta_k x_k(0)\\ &+ \xi_k x_k(t)) - \lambda\mathfrak{F}\}\rangle_{\text{Av}}\end{aligned} \tag{9}$$

where λ is taken to be real if $\mathfrak{F} > 0$, and negative imaginary otherwise.

Proceeding as in Parts I and II, we then obtain

$$\frac{\partial \hat{r}}{\partial t} = \sum_{j,l}\left(\frac{\partial \hat{r}}{\partial \xi_j} + \xi_j\hat{r}\right)Q_{jl}\xi_l + \lambda \sum_{j,l} K_{jl}(t)\frac{\partial^2 \hat{r}}{\partial \xi_j\, \partial \xi_l}. \tag{10}$$

The initial condition is obtained from (9) as

$$\hat{r}_{t=0} = \exp\{-\tfrac{1}{2}\sum_k (\eta_k + \xi_k)^2\}. \tag{11}$$

Eq. (10) is reduced to a set of first-order differential equations with t as the independent variable by the ansatz

$$\hat{r} = fe^{\phi} \tag{12}$$

where f is a function of t alone, and

$$\phi \equiv -\tfrac{1}{2}\sum_{i,k}[\eta_i a_{ik}(t)\eta_k + 2\eta_i b_{ik}(t)\xi_k + \xi_i c_{ik}(t)\xi_k] \tag{13}$$

where a and c are symmetric matrices depending only on t. Eq. (10) then becomes

$$\begin{aligned}\frac{\dot{f}}{f} + \dot{\phi} = &\sum_{j,l}\left(\frac{\partial \phi}{\partial \xi_j} + \xi_j\right)Q_{jl}\xi_l\\ &+ \lambda\sum_{j,l} K_{jl}\left(\frac{\partial^2 \phi}{\partial \xi_j\, \partial \xi_l} + \frac{\partial \phi}{\partial \xi_j}\frac{\phi}{\partial \xi_l}\right)\end{aligned} \tag{14}$$

[4] Some of these relations could be shown to be independent of the pecial form of the kernel. Recently, R. Bellman obtained a Riccati ype integrodifferential equation for the Fredholm resolvent of a redholm integral equation; see, "Functional Equations in the heory of Dynamic Programming-VII: An Integro-Differential quation for the Fredholm Resolvent," The RAND Corp., Santa Monica, Calif., Paper P-859; May 7, 1956.

[5] M. C. Wang and G. E. Uhlenbeck, "On the theory of the Brownian Motion II," *Rev. Mod. Phys.*, vol. 17, pp. 323–342; August, 1945.

[6] This can be assumed without loss of generality; see *ibid*, footnote 19, p. 330.

where dots indicate differentiation with respect to t. Using the symmetry of the matrix c, one has

$$\frac{\partial \phi}{\partial \xi_j} = -\sum_i \eta_i b_{ij} - \tfrac{1}{2} \sum_i (\xi_i c_{ij} + c_{ji}\xi_i)$$
$$= -\sum_i (\eta_i b_{ij} + \xi_i c_{ij}) \quad (15)$$

and (14) becomes

$$d \ln f/dt - \tfrac{1}{2} \sum_{i,k} (\eta_i \dot{a}_{ik} \eta_k + 2\eta_i \dot{b}_{ik} \xi_k + \xi_i \dot{c}_{ik} \xi_k) \quad (16)$$
$$= -\sum_{j,l} \Big(\sum_i (\eta_i b_{ij} + \xi_i c_{ij}) - \xi_j\Big) Q_{jl}$$
$$+ \lambda \sum_{j,l} K_{jl}\Big[-c_{lj} + \sum_{ik} (\eta_i b_{ij} + \xi_i c_{ij})(\eta_k b_{kl} + \xi_k c_{kl})\Big]$$
$$= -\lambda \sum_{jl} c_{lj} K_{jl} + \lambda \sum_{i,j,k,l} \eta_i (b_{ij} K_{jl} b_{lk}) \eta_k$$
$$+ \sum_{i,k} \eta_i\Big[-\sum_j b_{ij} Q_{jk} + 2\lambda \sum_{jl} b_{ij} K_{jl} c_{lk}\Big]\xi_k$$
$$+ \sum_{i,k} \xi_i\Big[\Big(-\sum_j c_{ij} Q_{jk} + Q_{ik}\Big) + \lambda \sum_{jl} c_{ij} K_{jl} c_{lk}\Big]\xi_k$$
$$= -\lambda \operatorname{tr}(Kc) + \lambda \sum_{i,k} \eta_i (bKb)_{ik} \eta_k$$
$$+ \sum_{ik} \eta_i [-(bQ)_{ik} + 2\lambda (bKc)_{ik}]\xi_k$$
$$+ \sum_{i,k} \xi_i [-(cQ)_{ik} + Q_{ik} + \lambda (cKc)_{ik}]\xi_k$$
$$= -\lambda \operatorname{tr}(Kc) + \frac{\lambda}{2} \sum_{i,k} \eta_i (bKb + \tilde{b}K\tilde{b}) \eta_k$$
$$+ \sum_{i,k} \eta_i [-(bQ)_{ik} + 2\lambda (bKc)_{ik}]\xi_k$$
$$+ \tfrac{1}{2} \sum_{i,k} \xi_i (-cQ - \tilde{Q}c + Q + \tilde{Q} + 2\lambda cKc)_{ik} \xi_k$$

where the second and fourth terms have been written in symmetric form in order to facilitate comparison of coefficients, and where tr indicates the trace of a matrix. One thus obtains

$$\frac{d \ln f}{dt} = -\lambda \operatorname{tr}(Kc) \quad (17a)$$
$$\dot{c} = -(Q + \tilde{Q}) + cQ + \tilde{Q}c - 2\lambda cKc \quad (17b)$$
$$\dot{a} = -\lambda(bKb + \tilde{b}K\tilde{b}) \quad (17c)$$
$$\dot{b} = b[Q - 2\lambda Kc]. \quad (17d)$$

The initial conditions are obtained from (11)–(13) as

$$f(0) = 1 \quad (18a)$$
$$a(0) = b(0) = c(0) = I, \quad (18b)$$

where I denotes the unit matrix.

The scalar equation (17a) and the matrix differential equations (17b)–(17d) reduce to (14a)–(14d) of Part II for the one-dimensional case. Eqs. (17a) and (17c) can still be solved by quadratures, but (17d) is no longer as simple as the corresponding (14d) of Part II. It turns out, however, that the solution of (17d) depends on the solution of the same set of linear equations as (17b) and therefore we will start with the latter.

Eq. (17b) is a matrix analog of the scalar Riccati equation, but the familiar substitution leading from the latter to a second-order linear differential equation does not work for the matrix equation. A similar substitution, however, leads to two linear differential equations for two matrices u and v. We write c in the form

$$c = vu^{-1} \quad (19)$$

and obtain

$$\dot{v}u^{-1} - vu^{-1}\dot{u}u^{-1} = -(Q + \tilde{Q}) + vu^{-1}Q + \tilde{Q}vu^{-1} - 2\lambda vu^{-1}Kvu^{-1} \quad (20)$$

or

$$\dot{v} = \tilde{Q}v - (Q + \tilde{Q})u + vu^{-1}(\dot{u} + Qu - 2\lambda Kv). \quad (21)$$

Since we are free to choose one relation between u and v, we can let the factor of vu^{-1} be zero and obtain the two linear differential equations

$$\dot{v} = \tilde{Q}v - (Q + \tilde{Q})u \quad (22)$$
$$\dot{u} = 2\lambda Kv - Qu. \quad (23)$$

The initial condition is $u(0) = v(0)$, since $c(0) = I$. The matrix Riccati equation (17b) is thus reduced to a pair of first-order linear differential equations for the matrices u and v, *i.e.*, to a system of $2n$ first-order linear differential equations.

The matrix b turns out to be simply

$$b(t) = u(0)u^{-1}(t), \quad (24)$$

since (17d), (19), and (23) yield

$$\dot{b} = -b\dot{u}u^{-1} \quad (25)$$

or

$$\dot{b}u + b\dot{u} = 0 \quad (26)$$

and, therefore,

$$bu = b(0)u(0) = u(0). \quad (27)$$

The function f also is expressed simply in terms of the determinant of u

$$\frac{d}{dt} \det u = \sum_{ik} \frac{du_{ik}}{dt} \frac{\partial \det u}{\partial u_{ik}} \quad (28)$$
$$= \sum_{ik} \frac{du_{ik}}{dt} (u^{-1})_{ik} \det u,$$

where $\det u$ denotes the determinant of u. We thus have

$$\frac{d \ln \det u}{dt} = \operatorname{tr} \dot{u}u^{-1} = \operatorname{tr}(2\lambda Kvu^{-1} - Q) \quad (29)$$
$$= -\operatorname{tr} Q + 2\lambda \operatorname{tr} Kc$$

and

$$\frac{d \ln f}{dt} = -\frac{1}{2} \operatorname{tr} Q - \frac{1}{2} \frac{d}{dt} \ln \det u. \tag{30}$$

Since $f(0) = 1$, we thus have

$$f(t) = e^{-t \operatorname{tr} Q/2} \sqrt{\frac{\det u(0)}{\det u(t)}}. \tag{31}$$

The matrix $u(0)$ can be chosen arbitrarily, as long as the matrix $v(0)$ is chosen equal to $u(0)$.

One could have obtained (22)–(24) and (19) directly, using (17d) to obtain

$$\frac{db^{-1}}{dt} = -b^{-1}\dot{b}b^{-1} = -(Qb^{-1} - 2\lambda Kcb^{-1}) \tag{32}$$

and (17b) to obtain

$$\begin{aligned} \frac{d(cb^{-1})}{dt} &= \dot{c}b^{-1} + c\frac{db^{-1}}{dt} \qquad (33) \\ &= -(Q + \tilde{Q})b^{-1} + cQb^{-1} + \tilde{Q}cb^{-1} \\ &\qquad - 2\lambda cKcb^{-1} - c(Qb^{-1} - 2\lambda Kcb^{-1}) \\ &= -(Q + \tilde{Q})b^{-1} + \tilde{Q}cb^{-1} \end{aligned}$$

so that one has two linear equations for b^{-1} and cb^{-1}.

III. The Case of Constant Matrix K

This case is of interest since the characteristic function of $\mathfrak{F}$ defined by (8), is for constant K, closely related to the characteristic function of the empirical variances and covariances of the components of $X(\tau)$.

The solution then can be reduced to the solution of $2n$ homogeneous linear algebraic equations, by writing (22) and (23) in the form

$$\begin{pmatrix} v \\ u \end{pmatrix} = \mathfrak{M} \begin{pmatrix} v \\ u \end{pmatrix} \tag{34}$$

where $\mathfrak{M}$ is the $2n \times 2n$ matrix

$$\mathfrak{M} = \begin{pmatrix} \tilde{Q} & -(Q + \tilde{Q}) \\ 2\lambda K & -Q \end{pmatrix}. \tag{35}$$

(Script capitals will be used to denote the $2n \times 2n$ matrices.)

Choosing the initial conditions $u(0) = v(0) = I$, we then have

$$\begin{pmatrix} v \\ u \end{pmatrix} = e^{\mathfrak{M}t} \begin{pmatrix} I \\ I \end{pmatrix} \tag{36}$$

and

$$u = (e^{\mathfrak{M}t})_{21} + (e^{\mathfrak{M}t})_{22} \tag{37}$$

$$e^{\mathfrak{M}t} = \begin{pmatrix} (e^{\mathfrak{M}t})_{11} & (e^{\mathfrak{M}t})_{12} \\ (e^{\mathfrak{M}t})_{21} & (e^{\mathfrak{M}t})_{22} \end{pmatrix}. \tag{38}$$

where $(e^{\mathfrak{M}t})_{21}$ and $(e^{\mathfrak{M}t})_{22}$ are $n \times n$ submatrices of $e^{\mathfrak{M}t}$, obtained by dividing $e^{\mathfrak{M}t}$ in the form

The unconditional characteristic function $f(t)$ for the distribution of $\mathfrak{F}(t)$ is obtained thus from (31) as

$$f(t) = e^{-t \operatorname{tr} Q/2} [\det \{(e^{\mathfrak{M}t})_{21} + (e^{(\mathfrak{M}t)})_{22}\}]^{-1/2}. \tag{39}$$

If the matrix $\mathfrak{M}$ can be diagonalized,

$$\mathfrak{M} = \mathfrak{C}^{-1}\mathfrak{D}\mathfrak{C}, \tag{40}$$

where $\mathfrak{D}$ is a diagonal matrix, the matrix $e^{\mathfrak{M}t}$ can be evaluated by using

$$e^{\mathfrak{M}t} = \mathfrak{C}^{-1}e^{\mathfrak{D}t}\mathfrak{C}, \tag{41}$$

where $e^{\mathfrak{D}t}$ is the diagonal matrix with elements $\exp(\mathfrak{D}_{ii}t)$.

Alternatively, one can evaluate $e^{\mathfrak{M}t}$ by the following procedure, which does not depend on the possibility of diagonalizing $\mathfrak{M}$. Let $D(z)$ be the secular determinant of $\mathfrak{M}$,

$$D(z) = \det(zI - \mathfrak{M}) \tag{42}$$

which is a polynomial of degree $2n$ in z. Define the function $\phi(\mu, z)$ by

$$\phi(\mu, z) = \frac{D(\mu) - D(z)}{\mu - z}. \tag{43}$$

[The explicit form of $\phi(\mu, z)$ is given in (49).]

Since $(\mu - z)$ is a divisor of $D(\mu) - D(z)$, the function $\phi(\mu, z)$ is a polynomial of degree $(2n - 1)$ in both μ and z, and is symmetric in μ and z. We then have

$$e^{\mathfrak{M}t} = \frac{1}{2\pi i} \oint e^{\mu t} \frac{\phi(\mu, \mathfrak{M})}{D(\mu)} d\mu \tag{44}$$

where the path of integration encircles all roots of $D(\mu)$, and the integral can be evaluated by the method of residues and appears as a polynomial of degree $2n - 1$ in the matrix $\mathfrak{M}$ with time-dependent coefficients. The fact that $e^{\mathfrak{M}t}$ has this particular form can already be concluded from the Cayley[7] theorem, which states that

$$D(\mathfrak{M}) = 0 \tag{45}$$

and allows expression of $\mathfrak{M}^{2n}$ and all higher powers of $\mathfrak{M}$ in the power series for $e^{\mathfrak{M}t}$ by the first $2n - 1$ powers of $\mathfrak{M}$.

That the coefficients are correctly given by (44) can be seen by differentiating (44). One obtains

$$\begin{aligned} \left(\frac{d}{dt} - \mathfrak{M}\right)e^{\mathfrak{M}t} &= \frac{1}{2\pi i} \oint e^{\mu t} \frac{(\mu - \mathfrak{M})\phi(\mu, \mathfrak{M})}{D(\mu)} d\mu \qquad (46) \\ &= \frac{1}{2\pi i} \oint e^{\mu t} d\mu = 0. \end{aligned}$$

To complete the proof, one need only to show that (44) yields the unit matrix for $t = 0$. We transform to a new variable $\xi = 1/\mu$ and can contract the path of integration to a small circle around $\xi = 0$, and obtain

$$\frac{1}{2\pi i} \oint^{0,+} \frac{\phi(\mu, \mathfrak{M})}{D(\mu)} d\mu = \frac{1}{2\pi i} \oint^{0,+} \frac{\phi(\xi^{-1}, \mathfrak{M})}{\xi^2 D(\xi^{-1})} d\xi. \tag{47}$$

[7] R. Courant and D. Hilbert, "Methoden der Mathematischen Physik," Springer Verlag, Berlin, Germany, vol. 1, p. 18; 1931.

Now if

$$D(z) = \sum_{s=0}^{2n} b_s z^s, \tag{48}$$

then

$$\begin{aligned}\phi(\mu, z) &= \sum_{s=1}^{2n} b_s \frac{\mu^s - z^s}{\mu - z} \\ &= \sum_{s=1}^{2n} b_s \sum_{r=0}^{s-1} \mu^r z^{s-1-r} \\ &= \sum_{r=0}^{2n-1} \mu^r \sum_{s=r+1}^{2n} b_s z^{s-1-r}\end{aligned} \tag{49}$$

and

$$\begin{aligned}\oint^{0,+} \frac{\phi(\xi^{-1}, \mathfrak{M})}{\xi^2 D(\xi^{-1})}\, d\xi &= \oint^{0,+} \frac{\sum_{r=0}^{2n-1} \xi^{-r} \sum_{s=r+1}^{2n} b_s \mathfrak{M}^{s-1-r}}{\xi^2 \sum_{s=0}^{2n} b_s \xi^{-s}} \\ &= \oint^{0,+} \frac{d\xi}{\xi} \frac{b_{2n} I + \xi(b_{2n}\mathfrak{M} + b_{2n-1} I) + \cdots}{b_{2n} + \xi b_{2n-1} + \cdots} \\ &= 2\pi i I.\end{aligned}$$

IV. Reduction of a Special Case to the One-Dimensional Problem

If Q is symmetric (which implies real eigenvalues), it can be written in the form

$$Q = \tilde{Q} = S^{-1} \Lambda S \tag{50}$$

where Λ is a diagonal matrix, since by definition Q is real. It turns out that the matrix S need not be known explicitly. Eqs. (22) and (23) then can be reduced to

$$\dot{V} = \Lambda V - 2\Lambda U \tag{51}$$

$$\dot{U} = 2\lambda \bar{K} V - \Lambda U \tag{52}$$

where

$$V = Sv, \qquad U = Su, \qquad \bar{K} = SKS^{-1}. \tag{53}$$

Especially if K commutes with Q, the matrix S can be chosen so that $\bar{K}$ is diagonal and, if its elements do not vanish identically,[8] (51) and (52) reduce to those of the one-dimensional case. If $\bar{u}$ is defined by

$$\bar{u} = e^{\Lambda t} U, \tag{54}$$

then

$$\dot{\bar{u}} = e^{\Lambda t}(\dot{U} + \Lambda V) = e^{\Lambda t} 2\lambda \bar{K} V. \tag{55}$$

Since $\bar{K}$ was assumed to be diagonal, we then have

$$\begin{aligned}\left(-\frac{1}{2}\Lambda^{-1}\frac{d}{dt} + 1\right)[(2\lambda\bar{K})^{-1}\dot{\bar{u}}] &= \left(-\frac{1}{2}\Lambda^{-1}\frac{d}{dt} + 1\right)(e^{\Lambda t} V) \\ &= e^{\Lambda t} U.\end{aligned} \tag{56}$$

[8] If some elements of $\bar{K}$ vanish, the corresponding rows of U are obtained directly from (52) and the initial condition as $U_{jk} = e^{\Lambda_{jj} t}\delta_{jk}$, otherwise the calculation is unchanged.

We thus obtain, with

$$\Lambda_{ii} = -\beta_i \tag{57}$$

and

$$\bar{K}_{ii} = \kappa_i(t), \tag{58}$$

the equation

$$\frac{1}{2}\beta_i^{-1}\frac{d}{dt}\left(\frac{1}{2\lambda\kappa_i(t)}\dot{\bar{u}}_{ik}\right) + \frac{1}{2\lambda\kappa_i(t)}\dot{\bar{u}}_{ik} - \bar{u}_{ik} = 0 \tag{59}$$

or with

$$dx_i = 2\lambda\kappa_i(t)\, dt \tag{60}$$

the equation

$$\lambda\beta_i^{-1} k_i(t(x_i))\, d^2\bar{u}_{ik}/dx_i^2 + d\bar{u}_{ik}/dx_i - \bar{u}_{ik} = 0 \tag{61}$$

which are the same as (21) of Part II, and were solved there explicitly for some special forms of $k_i(t)$.

The initial condition $u(0) = v(0)$ requires only $U(0) = V(0)$ and the initial condition for (61) is, therefore,

$$\left(\frac{d\bar{u}}{dt}\right)_{t=0} = 2\lambda\bar{K}(0)\bar{u}_{t=0}. \tag{62}$$

If $\kappa_i(0)$ does not vanish, this simplifies to

$$\left[\frac{d\bar{u}_{ik}}{dx_i}\right]_{t=0} = [u_{ik}]_{t=0}. \tag{63}$$

Without losing generality, one can choose

$$[\bar{u}_{t=0}] = I; \tag{64}$$

then $\bar{u}$ is a diagonal matrix for all x [provided that $\kappa_i(0) < \infty$] and one has

$$f(t) = \left\{\prod_{j=1}^{n} \bar{u}_{jj}(x(t))\right\}^{-1/2}$$

with $\bar{u}_{jj}(x)$ defined by (61), (63), and (64).

V. Derivation of the Matrix Riccati Equation by the Older Method

Since $R_{ii}(\tau)$ is the autocorrelation function of $x_i(t)$, it can be written in the form

$$R_{ii}(\tau) = \int_{-\infty}^{\infty} h_i(\vartheta) h_i(\tau + \vartheta)\, d\vartheta; \tag{65}$$

where one may demand $h_i(\vartheta) = 0$ for $\vartheta < 0$. One then can write $x_i(t)$ in the form

$$x_i(t) = \sum_\nu c_\nu \int_{-\infty}^{\infty} h_i(t - \vartheta)\psi_\nu(\vartheta)\, d\vartheta, \tag{66}$$

where the coefficients c_ν are independent Gaussian random variables with mean zero and variance unity, and where the functions $\psi_\nu(\vartheta)$ are a complete orthonormal set. Then the correlation matrix is given by

$$R_{kl}(\tau) = \int_{-\infty}^{\infty} h_k(\vartheta) h_l(\tau + \vartheta)\, d\vartheta. \tag{67}$$

Now let the functions $\psi_\nu(t)$ be the eigenfunctions of the integral equation

$$\lambda_\nu \psi_\nu(\vartheta_1) = \int_{-\infty}^{\infty} \Lambda(\vartheta_1, \vartheta_2)\psi_\nu(\vartheta_2)\, d\vartheta_2, \tag{68}$$

where

$$\Lambda(\vartheta_1, \vartheta_2) = \int_0^t \sum_{j,l} h_j(\vartheta - \vartheta_1)K_{jl}(\vartheta)h_l(\vartheta - \vartheta_2)\, d\vartheta; \tag{69}$$

then we have

$$\sum_{jl} \int_0^t x_j(\vartheta)K_{jl}(\vartheta)x_l(\vartheta)\, d\vartheta = \sum_\nu c_\nu^2\lambda_\nu, \tag{70}$$

using (66), (68), and (69).

One then obtains for the joint characteristic function

$$\begin{aligned} &\left\langle \exp\left[i \sum_j (\eta_j x_j(0) + \xi_j x_j(t)) - \lambda \int_0^t \sum_{j,l} K_{jl}(\tau)x_j(\tau)x_l(\tau)\, d\tau \right]\right\rangle_{\text{Av}} \\ &= \left\langle \exp\left[i \sum_\nu c_\nu \Psi_\nu - \lambda \sum_\nu c_\nu^2 \lambda_\nu \right]\right\rangle_{\text{Av}} \\ &= \prod_\nu (1 + 2\lambda\lambda_\nu)^{-1/2} \cdot \exp\left[-\frac{1}{2} \sum_\nu \Psi_\nu^2/(1 + 2\lambda\lambda_\nu)\right] \end{aligned} \tag{71}$$

with the abbreviations

$$\Psi_\nu \equiv \sum_j [\eta_j \phi_{j\nu}(0) + \xi_j \phi_{j\nu}(t)] \tag{72}$$

and

$$\phi_{j\nu}(t) \equiv \int_{-\infty}^{\infty} h_j(t - \vartheta)\psi_\nu(\vartheta)\, d\vartheta. \tag{73}$$

The function f and the matrix c introduced by (12) and (13), therefore, are given by

$$f = \prod_\nu (1 + 2\lambda\lambda_\nu)^{-1/2} \tag{74}$$

and

$$c_{kl} = \sum_\nu \frac{\phi_{k\nu}(t)\phi_{l\nu}(t)}{1 + 2\lambda\lambda_\nu}. \tag{75}$$

We now define the matrix function $g(\tau_1, \tau_2)$ by

$$\begin{aligned} g_{kl}(\tau_1, \tau_2) &= \sum_\nu \frac{\phi_{k\nu}(\tau_1)\phi_{l\nu}(\tau_2)}{1 + 2\lambda\lambda_\nu} \\ &= \iint_{-\infty}^{\infty} h_k(\tau_1 - \vartheta_1)h_l(\tau_2 - \vartheta_2)\gamma(\vartheta_1, \vartheta_2)\, d\vartheta_1 d\vartheta_2, \end{aligned} \tag{76}$$

where

$$\gamma(\vartheta_1, \vartheta_2) \equiv \sum_\nu \frac{\psi_\nu(\vartheta_1)\psi_\nu(\vartheta_2)}{1 + 2\lambda\lambda_\nu}. \tag{77}$$

Using (68), one sees that

$$\gamma(\vartheta_1, \vartheta_2) + 2\lambda \int_{-\infty}^{\infty} \Lambda(\vartheta_1, \vartheta)\gamma(\vartheta, \vartheta_2)\, d\vartheta = \delta(\vartheta_1 - \vartheta_2) \tag{78}$$

and, therefore,

$$\begin{aligned} g_{il}(\tau_1, \tau_2) &+ 2\lambda \iiint h_i(\tau_1 - \vartheta_1)\Lambda(\vartheta_1, \vartheta)\gamma(\vartheta, \vartheta_2) \cdot h_l(\tau_2 - \vartheta_2)\, d\vartheta\, d\vartheta_1\, d\vartheta_2 \\ &= \int h_i(\tau_1 - \vartheta_1)h_l(\tau_2 - \vartheta_1)\, d\vartheta_1 \\ &= g_{il}(\tau_1, \tau_2) \\ &+ 2\lambda \int h_i(\tau_1 - \vartheta_1) \int_0^t \sum_{jk} h_j(\tau - \vartheta_1)K_{jk}(\tau) \cdot h_k(\tau - \vartheta)\gamma(\vartheta, \vartheta_2)h_l(\tau_2 - \vartheta_2)\, d\tau\, d\vartheta\, d\vartheta_1\, d\vartheta_2 \\ &= g_{il}(\tau_1, \tau_2) \\ &+ 2\lambda \int_0^t \sum_{jk} R_{ij}(\tau - \tau_1)K_{jk}(\tau)g_{kl}(\tau, \tau_2) \\ &= R_{il}(\tau_2 - \tau_1), \end{aligned} \tag{79}$$

using (76), (69), and (65).

In matrix form this equation is written as

$$g(\tau_1, \tau_2) + 2\lambda \int_0^t R(\tau - \tau_1)K(\tau)g(\tau, \tau_2)d\tau = R(\tau_2 - \tau_1) \tag{80}$$

and $g(t, t)$ is the matrix $c(t)$.

The iteration solution, written for $\tau_1 = \tau_2 = t$, becomes

$$g(t, t) = I + \sum_1^\infty (-2\lambda)^n \int_0^t H^{(n)}(t, \tau)R(t - \tau)\, d\tau \tag{81}$$

where

$$H^{(1)}(\tau_1, \tau) \equiv H(\tau_1, \tau) = R(\tau - \tau_1)K(\tau) \tag{82}$$

and

$$H^{(n)}(\tau_1, \tau_2) = \int_0^t H^{(n-1)}(\tau_1, \tau)H^{(1)}(\tau, \tau_2)\, d\tau. \tag{83}$$

Differentiation yields

$$\begin{aligned} \frac{dg(t, t)}{dt} = \sum_1^\infty (-2\lambda)^n \Big\{ H^{(n)}(t, t) &+ \int_0^t \left[\frac{dH^{(n)}(t, \tau)}{dt} R(t - \tau) \right. \\ &\left. + H^{(n)}(t, \tau)R(t - \tau)Q \right] d\tau \Big\}, \end{aligned} \tag{84}$$

where

$$\begin{aligned} \frac{dH^{(n)}(t, \tau)}{dt} &= \frac{d}{dt} \int_0^t \cdots \int_0^t H(t, \tau_1)H(\tau_1, \tau_2) \cdots H(\tau_{n-1}, \tau)\, d\tau_1 \cdots d\tau_{n-1} \\ &= \sum_{k=1}^{n-1} H^{(k)}(t, t)H^{(n-k)}(t, \tau) \\ &+ \int_0^t \frac{H(t, \tau_1)}{\partial t} H^{(n-1)}(\tau_1, \tau)\, d\tau_1. \end{aligned} \tag{85}$$

Using (82), (4), and (5), one has

$$\frac{\partial H(t, \tau)}{\partial t} = \tilde{Q}H(t, \tau). \tag{86}$$

Substituting in (84) and using (81), one obtains

$$\frac{dg(t, t)}{dt} = \tilde{Q}(g(t, t) - I) + (g(t, t) - I)Q + \sum_{1}^{\infty} (-2\lambda)^n \Big\{ H^{(n)}(t, t) + \sum_{k=1}^{n-1} H^{(k)}(t, t) \int_0^t H^{(n-k)}(t, \tau)R(t - \tau)\, d\tau \Big\}. \tag{87}$$

It is convenient to define

$$\int_0^t H^{(0)}(t, \tau)\phi(\tau)\, d\tau = \phi(t) \tag{88}$$

for any function $\phi(\tau)$; then the last sum can be written as

$$\begin{aligned} &\sum_{n=1}^{\infty} (-2\lambda)^n \sum_{k=1}^{n} H^{(k)}(t, t) \int_0^t H^{(n-k)}(t, \tau)R(t - \tau)\, dt \\ &= \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} (-2\lambda)^{k+l} H^{(k)}(t, t) \int_0^t H^{(l)}(t, \tau)R(t - \tau)\, dt \\ &= \sum_{m=0}^{\infty} (-2\lambda)^{m+1} H^{(m+1)}(t, t) \cdot \sum_{l=0}^{\infty} (-2\lambda)^l \int_0^t H^{(l)}(t, \tau)R(t - \tau)\, d\tau \\ &= -2\lambda \cdot \Big[K(t) + \sum_{m=1}^{\infty} (-2\lambda)^m H^{(m+1)}(t, t) \Big] \cdot \Big[I + \sum_{l=1}^{\infty} (-2\lambda) \int_0^t H^{(l)}(t, t)R(t - \tau)\, d\tau \Big] \end{aligned} \tag{89}$$

using (82) and (88). We recognize the second bracket as $g(t, t)$ and the first bracket as $g(t, t)\ K(t)$, since

$$R(t - \tau)K(t) = H(\tau, t) \tag{90}$$

and therefore

$$g(t, t)K(t) = K(t) + \sum_{n=1}^{\infty} (-2\lambda)^n \int_0^t H^{(n)}(t, \tau)H(\tau, t)\, d\tau. \tag{91}$$

Now writing $c(t)$ for $g(t, t)$ we obtain from (87) and (89)

$$\frac{dc(t)}{dt} = -\tilde{Q} - Q + \tilde{Q}c(t) + c(t)Q - 2\lambda c(t)K(t)c(t) \tag{92}$$

in agreement with (17b).

VI. Characteristic Functions for the Probability Distribution of $\int_0^\infty e^{-t}\, x_1^2\, (t)\, dt$

For the calculations of this section it is useful to have (22), (23), and (31) in a different form.

Since

$$e^{-t \operatorname{tr} Q} = \det e^{-Qt}, \tag{93}$$

(31) can be written as

$$f^{-2}(t) = \det\, [e^{Qt}u(t)] \tag{94}$$

when $u(0) = v(0)$ is chosen to be the unit matrix.

From (23) it follows that

$$\frac{d}{dt}\, [e^{Qt}u(t)] = 2\lambda e^{Qt}K(t)v(t) \tag{95}$$

and

$$e^{Qt}u(t) = I + 2\lambda \int_0^t e^{Qt'}K(t')v(t')\, dt'. \tag{96}$$

This equation is specially useful in the limit $t \to \infty$, since the integral, in this limit, can be expressed in terms of the Laplace transform of $K(t')\ v(t')$ with $-\ Q$ substituted for the Laplace variable.

With

$$V_{hl}(t) \equiv [K(t)v(t)]_{hl} \tag{97}$$

and

$$\hat{V}_{hl}(s) \equiv \int_0^\infty e^{-st}V_{hl}(t)\, dt, \tag{98}$$

one has

$$\begin{aligned} \lim_{t\to\infty} [e^{Qt}u(t)]_{kl} &= \delta_{kl} + 2\lambda \sum_h \int_0^\infty (e^{Qt})_{kh}V_{hl}(t)\, dt \\ &= \delta_{kl} + 2\lambda \sum_h \int_0^\infty [e^{Qt}V_{hl}(t)]_{kh}\, dt \\ &= \delta_{kl} + 2\lambda \sum_h [\hat{V}_{hl}(-Q)]_{kh}. \end{aligned} \tag{99}$$

The derivation shows that the last expression is to be evaluated as follows. The matrix elements $\hat{V}_{hl}(s)$, which are ordinary functions of s, become matrices $\hat{V}_{hl}(-\ Q)$ when $-\ Q$ is substituted for s. These matrices have elements $[\hat{V}_{hl}(-\ Q)]_{kh}$, and the summation is to be carried out over the index h, to obtain the (kl) element of the matrix $\lim_{t\to\infty} (e^{Qt}\ u(t) - I)$. If the matrix Q is available in diagonalized form

$$Q_{kh} = -\sum_\gamma S_{k\gamma}\beta_\gamma S^{-1}_{\gamma h}, \tag{100}$$

the above result can be written as

$$\lim_{t\to\infty} [e^{Qt}u(t)]_{kl} = \delta_{kl} + 2\lambda \sum_\gamma S_{k\gamma}(S^{-1}\hat{V}(\beta_\gamma))_{\gamma l} \tag{101}$$

which can also be derived directly from (99). Another form of this result, which is useful in the special case treated below, is obtained from (44), with Q substituted for $\mathfrak{M}$. The integral in (99) then becomes

$$\int_0^\infty e^{Qt}V(t)\, dt = \int_0^\infty \frac{1}{2\pi i} \oint e^{\mu t} \frac{\varphi(\mu, Q)}{D(\mu)}\, V(t)\, dt = \frac{1}{2\pi i} \oint \frac{\varphi(\mu, Q)}{D(\mu)}\, \hat{V}(-\mu)\, d\mu \tag{99a}$$

where the path of integration encircles the roots of

$$D(\mu) \equiv \prod_{\kappa=1}^{n} (\mu + \beta_\kappa) \tag{99b}$$

and where $(-\beta_\kappa)$ are the n eigenvalues of Q. If these are all distinct, one has

$$\int_0^\infty e^{Qt} V(t)\, dt = \sum_{\gamma=1}^{n} \varphi(-\beta_\gamma, Q)\hat{V}(\beta_\gamma) \prod_{\substack{\kappa=1\\ \kappa\neq\gamma}}^{n} (\beta_\kappa - \beta_\gamma). \tag{99c}$$

The matrix $v(t)$ satisfies the integral equation

$$v(t) = e^{-Qt} + 2\lambda \int_0^t [e^{-Q(t-t')} - e^{\tilde{Q}(t-t')}]K(t')v(t')\, dt'. \tag{102}$$

To check this, one multiplies from the left by $e^{-\tilde{Q}t}$ and differentiates and obtains

$$\begin{aligned}\frac{d}{dt}[e^{-\tilde{Q}t}v(t)] &= \frac{d}{dt}[e^{-\tilde{Q}t}e^{-Qt}] + 2\lambda\frac{d}{dt}\left\{e^{-\tilde{Q}t}e^{-Qt}\int_0^t e^{Qt'}K(t')v(t')\, dt'\right\} - 2\lambda\frac{d}{dt}\int_0^t e^{-\tilde{Q}t'}K(t')v(t')\, dt' \\ &= \frac{d}{dt}[e^{-\tilde{Q}t}e^{-Qt}]\cdot\left(I + 2\lambda\int_0^t e^{Qt'}K(t')v(t')\, dt'\right),\end{aligned} \tag{103}$$

since the terms arising from differentiation of the integrals cancel. Using (23), one then has

$$\begin{aligned}\frac{d}{dt}[e^{-\tilde{Q}t}v(t)] &= \frac{d}{dt}[e^{-\tilde{Q}t}e^{-Qt}]\cdot e^{Qt}u(t) \\ &= e^{-\tilde{Q}t}(Q + \tilde{Q})u(t)\end{aligned} \tag{104}$$

in agreement with (22).

Instead of considering (102) as an integral equation for $v(t)$, one may consider its Laplace transform

$$\hat{v}(s) = (sI + Q)^{-1} + 2\lambda\{(sI + Q)^{-1} - (sI - \tilde{Q})^{-1}\}\hat{V}(s) \tag{105}$$

as an algebraic relation between $\hat{v}(s)$ and $\hat{V}(s)$ to be used for the determination of $\hat{v}(s)$ and $\hat{V}(s)$, together with

$$\hat{V}(s) = \int_0^\infty e^{-st}K(t)v(t)\, dt, \tag{106}$$

which follows from (97) and (98).

As an example of a case in which this approach is useful, we now consider the special case

$$K(t) = Re^{-t} \tag{107a}$$

where R is the constant matrix with elements

$$R_{kl} = \delta_{k1}\delta_{l1}. \tag{107b}$$

The case $K(t) = Re^{-\alpha t}$ of course, can be reduced to this case by choosing α^{-1} as unit of time. Eq. (106) then reduces to

$$\hat{V}(s) = R\hat{v}(s + 1) \tag{108}$$

so that only the elements of the first row of $\hat{v}(s)$ are needed for the evaluation of (99). For these, (105) reduces to the difference equation

$$\hat{v}_{1l}(s) = a_l(s) + 2\lambda b(s)\hat{v}_{1l}(s + 1) \tag{109}$$

where

$$a_l(s) \equiv (sI + Q)^{-1}_{1l} \tag{110}$$

$$b(s) \equiv \{(sI + Q)^{-1} - (sI - Q)^{-1}\}_{11}. \tag{111}$$

(The tilde indicating the transposed matrix can be omitted for the diagonal element.)

The functions $a_l(s)$ and $b(s)$ are rational functions of s.

Although there are more elegant methods of solving (109), the most transparent result is obtained by iteration, which yields

$$\hat{v}_{1l}(s) = a_l(s) + \sum_{r=1}^{\infty}(2\lambda)^r \prod_{m=0}^{r-1} b(s + m)\cdot a_l(s + r) \tag{112}$$

and

$$\hat{v}_{1l}(s + 1) = a_l(s + 1) + \sum_{r=1}^{\infty}(2\lambda)^r \prod_{m=1}^{r} b(s + m)a_l(s + 1 + r). \tag{113}$$

This can be written as

$$\hat{v}_{1l}(s + 1) = \sum_{r=0}^{\infty}(2\lambda)^r \prod_{m=1}^{r} b(s + m)a_l(s + 1 + r), \tag{113a}$$

if we adopt the convention that products containing no factors, such as $\prod_{m=1}^{0}$, are to be replaced by unity, in this and all subsequent equations.

The terms of the series can be obtained in a more explicit form by the procedure used in Section III, (42) *et seq.* Let $D(x)$ be the secular determinant of Q

$$D(x) = \det(xI - Q) = \prod_{\kappa=1}^{n}(x + \beta_\kappa) \tag{114}$$

when Q is a matrix of n rows and columns with eigenvalues $(-\beta_\gamma)$. The eigenvalues are assumed to be distinct. Let $\varphi(x, y)$ be the polynomial of degree $(n - 1)$ in both x and y defined by

$$(x - y)\varphi(x, y) = D(x) - D(y). \tag{115}$$

The matrices used in (110) and (111) then can be written in the form of polynomials in s and Q

$$(sI - Q)^{-1} = \varphi(s, Q)/D(s) \tag{116}$$

$$(sI + Q)^{-1} = -(-sI - Q)^{-1} = -\varphi(-s, Q)/D(-s). \tag{117}$$

The function $b(s)$ is thus

$$b(s) = -[\varphi(-s, Q)_{11}D(s) + \varphi(s, Q)_{11}D(-s)]/D(s)D(-s). \tag{118}$$

Since the terms with the highest exponent of x in $\varphi(x, y)$ and $D(x)$ are x^{n-1} and x^n, respectively, the coefficient of s^{2n-1} in the numerator vanishes and the numerator is a polynomial of degree $2(n - 1)$. It could be

of lower order only if Q_{11} vanished, which is a case of no practical interest. Only even powers of s can occur, because the numerator is obviously an even function of s.

Let the roots of the numerator in (118) be $\pm \alpha_\mu$ ($\mu = 1, 2, \cdots, n-1$). Then the function $b(s)$ can be written in the form

$$b(s) = \text{const} \cdot \prod_{\mu=1}^{n-1} (s^2 - \alpha_\mu^2) \Big/ \prod_{\kappa=1}^{n} (s^2 - \beta_\kappa^2). \tag{119}$$

The constant is easily determined by expanding (111) for large s. One has

$$b(s) = -2s^{-2}Q_{11} + \cdots \tag{120}$$

and, therefore,

$$b(s) = -2Q_{11} \prod_{\mu=1}^{n-1} (s^2 - \alpha_\mu^2) \Big/ \prod_{\kappa=1}^{n} (s^2 - \beta_\kappa^2). \tag{121}$$

In order to obtain the product in (112) explicitly, note that

$$\prod_{m=1}^{r} (s + m - \alpha) = \Gamma(s + r + 1 - \alpha)/\Gamma(s + 1 - \alpha) \tag{122}$$

and, therefore,

$$\prod_{m=1}^{r} b(s + m) = (-2Q_{11})^r \cdot \prod_{\mu=1}^{n-1} \frac{\Gamma(s + r + 1 - \alpha_\mu)\Gamma(s + r + 1 + \alpha_\mu)}{\Gamma(s + 1 - \alpha_\mu)\Gamma(s + 1 + \alpha_\mu)} \cdot \prod_{\kappa=1}^{n} \frac{\Gamma(s + 1 - \beta_\kappa)\Gamma(s + 1 + \beta_\kappa)}{\Gamma(s + r + 1 - \beta_\kappa)\Gamma(s + r + 1 + \beta_\kappa)} \tag{123}$$

where again for $n = 1$, the first product is to be replaced by unity. We need this expression only for $s = \beta_\gamma$, and can take out of the second product the term $\kappa = \gamma$

$$\prod_{m=1}^{r} b(\beta_\gamma + m) = \frac{(-2Q_{11})^r}{r!} \frac{\Gamma(1 + 2\beta_\gamma)}{\Gamma(1 + r + 2\beta_\gamma)} \cdot \prod_{\mu=1}^{n-1} \frac{\Gamma(\beta_\gamma + r + 1 - \alpha_\mu)\Gamma(\beta_\gamma + r + 1 + \alpha_\mu)}{\Gamma(\beta_\gamma + 1 - \alpha_\mu)\Gamma(\beta_\gamma + 1 + \alpha_\mu)} \cdot \prod_{\substack{\kappa=1 \\ \kappa \neq \gamma}}^{n} \frac{\Gamma(1 + \beta_\gamma - \beta_\kappa)\Gamma(1 + \beta_\gamma + \beta_\kappa)}{\Gamma(r + 1 + \beta_\gamma - \beta_\kappa)\Gamma(r + 1 + \beta_\gamma + \beta_\kappa)}. \tag{124}$$

The functions $a_l(\beta_\gamma + 1 + r)$ can be written as

$$\begin{aligned} a_l(\beta_\gamma + 1 + r) &= -\varphi(-\beta_\gamma - 1 - r, Q)_{1l}/D(-\beta_\gamma - 1 - r) \\ &= (-)^{n+1}\varphi(-\beta_\gamma - 1 - r, Q)_{1l} \Big/ \prod_{\kappa=1}^{n} (r + 1 + \beta_\gamma - \beta_\kappa) \end{aligned} \tag{125}$$

by use of (110), (114), and (117). The denominator of this expression combines with the factor

$$r! \prod_{\substack{\kappa=1 \\ \kappa \neq \gamma}}^{n} \Gamma(r + 1 + \beta_\gamma - \beta_\kappa)$$

in the preceding equation.

Eq. (99c) becomes, in the special case defined by (107a)–(108),

$$\int_0^\infty [e^{Qt}V(t)]_{kl}\, dt = \sum_{\gamma=1}^{n} \varphi(-\beta_\gamma, Q)_{k1}\hat{v}_{1l}(\beta_\gamma + 1) \Big/ \prod_{\substack{\kappa=1 \\ \kappa \neq \gamma}}^{n} (\beta_\kappa - \beta_\gamma). \tag{126}$$

Combining (113a) and (124)–(126), one obtains

$$\lim_{t\to\infty} [e^{Qt}u(t)]_{kl} = \delta_{kl} + 2\lambda \sum_{\gamma=1}^{n} \varphi(-\beta_\gamma, Q)_{k1} \cdot \sum_{r=0}^{\infty} \frac{(-4\lambda Q_{11})^r}{(r+1)!} \frac{\Gamma(1 + 2\beta_\gamma)}{\Gamma(1 + r + 2\beta_\gamma)} \varphi(-\beta_\gamma - 1 - r, Q)_{1l} \cdot \prod_{\mu=1}^{n-1} \frac{\Gamma(\beta_\gamma + r + 1 - \alpha_\mu)\Gamma(\beta_\gamma + r + 1 + \alpha_\mu)}{\Gamma(\beta_\gamma + 1 - \alpha_\mu)\Gamma(\beta_\gamma + 1 + \alpha_\mu)} \cdot \prod_{\substack{\kappa=1 \\ \kappa \neq \gamma}}^{n} \frac{\Gamma(\beta_\gamma - \beta_\kappa)\Gamma(1 + \beta_\gamma + \beta_\kappa)}{\Gamma(r + 2 + \beta_\gamma - \beta_\kappa)\Gamma(r + 1 + \beta_\gamma + \beta_\kappa)} \tag{127}$$

where φ is defined by (115), and the numbers $\pm \alpha_\mu$ are the roots of the numerator in (118), and $(-\beta_\gamma)$ are the eigenvalues of Q. A product without factors, such as $\prod_{\mu=1}^{n-1}$ with $n = 1$, has to be replaced by unity.

For $n = 1$, this reduces to the result obtained in (54) of Part II. In that case, the matrices reduce to numbers, and

$$Q = Q_{11} = -\beta_1 \tag{128}$$

$$D(x) = (x + \beta_1) \tag{129}$$

$$\varphi(x, y) = \frac{(x + \beta) - (y + \beta)}{x - y} = 1 \tag{130}$$

and, since the products contain no factors, the result reduces in the one-dimensional case to

$$\begin{aligned} f^{-2}(\infty) &= \lim_{t\to\infty} [e^{Qt}u(t)] \\ &= 1 + 2\lambda \sum_{r=1}^{\infty} \frac{(4\lambda\beta_1)^r}{(r+1)!} \frac{\Gamma(1 + 2\beta_1)}{\Gamma(1 + r + 2\beta_1)} \\ &= \Gamma(2\beta_1) \sum_{r=0}^{\infty} \frac{(4\lambda\beta_1)^r}{r!\Gamma(r + 2\beta_1)} \\ &= \Gamma(2\beta_1)(-4\lambda\beta_1)^{-\beta_1+1/2} J_{2\beta_1-1}(\sqrt{-16\lambda\beta_1}). \end{aligned} \tag{131}$$

With $\beta_1 = \beta/\alpha$ this agrees with the result quoted above.

If the diagonalizing matrix S defined by (100) is available, a matrix slightly simpler than that given in (127) can be obtained, which has the same determinant. Let the matrix F be defined by

$$F_{\sigma\tau} = \sum_{kl} S_{\sigma k}^{-1} \lim_{t\to\infty} [e^{Qt}u(t)]_{kl} S_{l\tau} \tag{132}$$

and note that the determinant is unchanged by this transformation. The same procedure applied to the matrix elements on the right-hand side of (127) results in

$$\sum_k S_{\sigma k}^{-1}\varphi(-\beta_\gamma, Q)_{k1} = \varphi(-\beta_\gamma, -\beta_\sigma)S_{\sigma 1}^{-1} \tag{133}$$

and

$$\sum_l \varphi(-\beta_\gamma - 1 - r, Q)_{1l}S_{l\tau} = S_{1\tau}\varphi(-\beta_\gamma - 1 - r, -\beta_\tau). \tag{134}$$

From (114) and (115) one obtains

$$\varphi(x, -\beta_\sigma) = D(x)/(x+\beta_\sigma) = \prod_{\substack{\kappa \\ \kappa\neq\sigma}} (x+\beta_\kappa), \quad (135)$$

$$\varphi(-\beta_\gamma, -\beta_\sigma) = \prod_{\substack{\kappa \\ \kappa\neq\sigma}} (\beta_\kappa - \beta_\gamma) = \delta_{\gamma\sigma} \prod_{\substack{\kappa \\ \kappa\neq\sigma}} (\beta_\kappa - \beta_\sigma) \quad (136)$$

and

$$\varphi(-\beta_\gamma - 1 - r, -\beta_\tau) = \prod_{\substack{\kappa \\ \kappa\neq\tau}} (\beta_\kappa - \beta_\gamma - 1 - r). \quad (137)$$

Substitution of these results in (132) yields

$$F_{\sigma\tau} = \delta_{\sigma\tau} + 2\lambda g_{\sigma\tau} S_{\sigma 1}^{-1} S_{1\tau} \quad (138)$$

with

$$g_{\sigma\tau} = \sum_{r=0}^{\infty} (-4\lambda Q_{11})^r \cdot \prod_{\mu=1}^{n-1} \frac{\Gamma(\beta_\sigma - \alpha_\mu + r + 1)\Gamma(\beta_\sigma + \alpha_\mu + r + 1)}{\Gamma(\beta_\sigma - \alpha_\mu + 1)\Gamma(\beta_\sigma + \alpha_\mu + 1)} \cdot \prod_{\kappa=1}^{n} \frac{\Gamma(\beta_\sigma - \beta_\kappa + 1)\Gamma(\beta_\sigma + \beta_\kappa + 1)}{\Gamma(\beta_\sigma - \beta_\kappa + r + 1)\Gamma(\beta_\sigma + \beta_\kappa + r + 1)} \cdot (\beta_\sigma - \beta_\tau + r + 1)^{-1}. \quad (139)$$

The series is of hypergeometric type.[9]

In the Appendix $(\det F)^{-1/2}$ has been computed to second order in λ and the result checked by direct computation of the first and second moment.

VII. Appendix

To check the result given by (138) and (139), we compute $f(\infty)$ to second order in λ from these equations and directly.

Writing the matrix F in the form

$$F = I + \lambda G + \lambda^2 H + \cdots \quad (140)$$

with

$$G_{\sigma\tau} = 2 S_{\sigma 1}^{-1} S_{1\tau} (\beta_\sigma - \beta_\tau + 1)^{-1} \quad (141)$$

$$H_{\sigma\tau} = 2 S_{\sigma 1}^{-1} S_{1\tau} (-4Q_{11}) \prod_{\mu=1}^{n-1} [(\beta_\sigma + 1)^2 - \alpha_\mu^2] \cdot \prod_{\kappa=1}^{n} [(\beta_\sigma + 1)^2 - \beta_\kappa^2]^{-1} (\beta_\sigma - \beta_\tau + 2)^{-1} \quad (142)$$

we have, to second order in λ,

$$\begin{aligned} f(\infty) &= (\det F)^{-1/2} = \exp\{-\tfrac{1}{2}\operatorname{tr}\ln F\} \\ &= \exp\{-\tfrac{1}{2}\operatorname{tr}[\lambda G + \lambda^2(H - \tfrac{1}{2}G^2)]\} \\ &= 1 - \frac{\lambda}{2}\operatorname{tr} G \\ &\quad - \frac{\lambda^2}{2}[\operatorname{tr} H - \tfrac{1}{2}\operatorname{tr} G^2 - \tfrac{1}{4}(\operatorname{tr} G)^2] + \cdots \end{aligned} \quad (143)$$

where tr stands for trace. The individual terms are evaluated as follows

$$\operatorname{tr} G = \sum_\sigma 2 S_{\sigma 1}^{-1} S_{1\sigma} = 2. \quad (144)$$

$$\begin{aligned} \operatorname{tr} G^2 &= \sum_{\sigma,\tau} G_{\sigma\tau} G_{\tau\sigma} \\ &= 4 \sum_{\sigma,\tau} S_{\sigma 1}^{-1} S_{1\tau} S_{\tau 1}^{-1} S_{1\sigma} (1 + \beta_\sigma - \beta_\tau)^{-1} \cdot (1 + \beta_\tau - \beta_\sigma)^{-1} \\ &= 2 \sum_{\sigma\tau} S_{\sigma 1}^{-1} S_{1\sigma} S_{\tau 1}^{-1} S_{1\tau} \left[\frac{1}{1+\beta_\sigma-\beta_\tau} + \frac{1}{1+\beta_\tau-\beta_\sigma}\right] \\ &= 2\Big(\sum_\sigma S_{\sigma 1}^{-1} S_{1\sigma} (1 + \beta_\sigma + Q)_{11}^{-1} + \sum_\tau S_{\tau 1}^{-1} S_{1\tau} (1 + \beta_\tau + Q)_{11}^{-1}\Big) \\ &= 4 \sum_\sigma S_{\sigma 1}^{-1} S_{1\sigma} a_1 (1 + \beta_\sigma) \\ &= 4[a_1(I - Q)]_{11} \end{aligned} \quad (145)$$

where $a_1(s)$ is defined by (110).

$$\begin{aligned} \operatorname{tr} H &= -4Q_{11} \sum_\sigma S_{\sigma 1}^{-1} S_{1\sigma} \cdot \prod_{\mu=1}^{n-1} [(\beta_\sigma + 1)^2 - \alpha_\mu^2] \prod_{\kappa=1}^{n} [(\beta_\sigma + 1)^2 - \beta_\kappa^2] \\ &= 2 \sum_\sigma S_{\sigma 1}^{-1} S_{1\sigma} b(\beta_\sigma + 1) \\ &= 2[b(I - Q)]_{11} \end{aligned} \quad (146)$$

where the explicit form of $b(s)$, given in (121), has been used. Substitution of these results in (143) yields

$$\begin{aligned} f(\infty) &= 1 - \lambda - \frac{\lambda^2}{2}\{2[b(I-Q)]_{11} - 2[a_1(I-Q)]_{11} - 1\} + \cdots \\ &= 1 - \lambda + \frac{\lambda^2}{2}\{1 + 2[a_1(I-Q) - b(I-Q)]_{11}\} + \cdots. \end{aligned} \quad (147)$$

The first two moments of

$$\mathfrak{F} \equiv \int_0^\infty e^{-t} x_1^2(t)\, dt, \quad (148)$$

where $x_1(t)$ is the first component of the stationary Gaussian process specified by (3)–(5) are easily obtained by direct computation.

The first moment is

$$\langle \mathfrak{F} \rangle_{\text{Av}} = \int_0^\infty e^{-t} \langle x_1^2(t) \rangle_{\text{Av}}\, dt = 1. \quad (149)$$

The second moment is

$$\begin{aligned} \langle \mathfrak{F}^2 \rangle_{\text{Av}} &= \left\langle \iint_0^\infty e^{-t-t'} x_1^2(t) x_1^2(t')\, dt\, dt' \right\rangle_{\text{Av}} \\ &= 2\int_0^\infty e^{-t} \int_0^t e^{-t'} \langle x_1^2(t') x_1^2(t) \rangle_{\text{Av}}\, dt'\, dt. \end{aligned} \quad (150)$$

[9] G. N. Watson, "A Treatise on the Theory of Bessel Functions," Cambridge University Press, New York, N. Y., p. 100; 1944.

For a Gaussian random function, one has

$$\langle x(t_1)x(t_2)x(t_3)x(t_4)\rangle_{Av} = \langle x(t_1)x(t_2)\rangle_{Av}\langle x(t_3)x(t_4)\rangle_{Av} + \langle x(t_1)x(t_3)\rangle_{Av}\langle x(t_2)x(t_4)\rangle_{Av} + \langle x(t_1)x(t_4)\rangle_{Av}\langle x(t_2)x(t_3)\rangle_{Av} \quad (151)$$

and, with $t_1 = t_2 = t'$ and $t_3 = t_4 = t$, and $t' \leq t$,

$$\langle x_1^2(t')x_1^2(t)\rangle_{Av} = \langle x_1^2(t')\rangle_{Av}\langle x_1^2(t)\rangle_{Av} + 2\langle x_1(t')x_1(t)\rangle_{Av}^2 = 1 + 2\{(e^{Q(t-t')})_{11}\}^2. \quad (152)$$

Substitution in (150) yields

$$\langle \mathfrak{F}^2\rangle_{Av} = 1 + 4\int_0^\infty e^{-t}\int_0^t e^{-t'}\{(e^{Q(t-t')})_{11}\}^2\, dt' = 1 + 2\int_0^\infty e^{-t}\{(e^{Qt})_{11}\}^2\, dt \quad (153)$$

by virtue of the folding theorem for Laplace transforms.

Using (100), one obtains

$$\langle \mathfrak{F}^2\rangle_{Av} = 1 + 2\sum_{\sigma,\tau} S_{1\sigma}S_{\sigma 1}^{-1}S_{1\tau}S_{\tau 1}^{-1}(1 + \beta_\sigma + \beta_\tau)^{-1} \quad (154)$$

or, with $a_1(s)$ and $b(s)$ defined by (110) and (111)

$$\langle \mathfrak{F}^2\rangle_{Av} = 1 + 2(a_1(I - Q) - b(I - Q))_{11}. \quad (155)$$

We, therefore, have to second order in λ

$$f(\infty) \equiv \langle e^{-\lambda\mathfrak{F}}\rangle_{Av} = 1 - \lambda\langle\mathfrak{F}\rangle_{Av} + \frac{\lambda^2}{2}\langle\mathfrak{F}^2\rangle_{Av} + \cdots = 1 - \lambda + \frac{\lambda^2}{2}\{1 + 2[a_1(I - Q) - b(I - Q)]_{11}\} + \cdots \quad (156)$$

in agreement with (147).

The Axis-Crossing Intervals of Random Functions—II*

J. A. McFADDEN†

Summary—This paper considers the intervals between axis crossings of a random function $\xi(t)$. Following a previous paper,[1] continued use is made of the statistical properties of the function $x(t)$ and the output after $\xi(t)$ is infinitely clipped. Under the assumption that a given axis-crossing interval is independent of the sum of the previous $(2m + 2)$ intervals, where m takes on *all* values, $m = 0, 1, 2, \cdots$, an integral equation is derived for the probability density $P_0(\tau)$ of axis-crossing intervals. This equation is solved numerically for several examples of Gaussian noise. The results of this calculation compare favorably with experiment when the high-frequency cutoff is not extremely sharp. Under the assumption that the successive axis-crossing intervals form a Markoff chain in the wide sense, infinite integrals are found which yield the variance $\sigma^2(\tau)$ and the correlation coefficient κ between the lengths of two successive axis-crossing intervals. These parameters are obtained numerically for several examples of Gaussian noise. For bandwidths at least as small as the mean frequency, κ is large. For low-pass spectra, κ is small, yet the statistical dependence between successive intervals may be strong even when the correlation κ is nearly zero.

Introduction

THE PROBLEM treated here is the determination of the distribution of intervals between axis crossings in random noise. In a previous paper,[1] the author discussed this distribution in terms of the autocorrelation function of infinitely clipped noise, without assuming that the original noise was Gaussian. This general discussion is continued now, and some new specific results are presented for the Gaussian case.

Wherever possible, the mathematical notation is the same as in the previous paper (I). The only exception is the use of the symbol $P_0(\tau)$.

In the next section, several infinite series are derived which involve $P_n(\tau)$, the probability density of the sum of $(n + 1)$ successive axis-crossing intervals. Following that, the initial behavior of $P_0(\tau)$ is discussed, particularly in the Gaussian case.

Next under consideration is the probability density $P_0(\tau)$ when successive axis-crossing intervals are independent, and again under a slightly weaker assumption, which is called quasi-independence. In either case, an integral equation must be solved numerically.

The last section solves for the variance of axis-crossing intervals and for the correlation coefficient between two successive intervals, under the assumption that the successive intervals form a Markoff chain in the wide sense.

Appendix I derives the general relation between $P_n(\tau)$ and $p(n, \tau)$, the probability that a given time interval of length τ contains exactly n zeros.

Derivation of the Basic Equations

Some infinite series will be derived now which will prove useful in the subsequent sections.

Let $\xi(t)$ describe a random process which is both stationary and ergodic. Assume that the mean value

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[1] J. A. McFadden, "The axis-crossing intervals of random functions," IRE Trans. on Information Theory, vol. 2, pp. 146–150; December, 1956. Hereafter, this reference will be denoted by the symbol (I).

$[\xi(t)] = 0$ and that the probability density of $\xi(t)$ is ymmetric about the mean. The normalized autocorrelaon function of $\xi(t)$ is denoted by $\rho(\tau)$.

Let the function $x(t)$ be the output after $\xi(t)$ is infinitely ipped. Then

$$x(t) = 1 \quad \text{if} \quad \xi(t) \geq 0; \qquad x(t) = -1 \quad \text{if} \quad \xi(t) < 0. \tag{1}$$

he autocorrelation function of $x(t)$ is denoted by $r(\tau)$. he function $x(t)$ is of interest because the axis crossings re preserved in the clipping process.

Now consider an interval of time $(t, t + \tau)$. Let $p(n, \tau)$ e the probability of finding exactly n zeros in the given nterval. If the number of zeros is even, then (and only hen) $x(t)$ and $x(t + \tau)$ have the same sign. Then the utocorrelation $r(\tau)$, or $E[x(t)x(t + \tau)]$, is given by the nfinite series,[2]

$$r(\tau) = \sum_{n=0}^{\infty} (-1)^n p(n, \tau). \tag{2}$$

Now define the function $P_n(\tau)$. $P_n(\tau)d\tau$ is the conitional probability that the $(n + 1)$st zero after time t ccurs between $t + \tau$ and $t + \tau + d\tau$, given a zero at me t. In other words, $P_n(\tau)$ is the probability density of he interval between the mth and $(m + n + 1)$st zeros, here m is arbitrary. $P_0(\tau)$ is the probability density of tervals between *successive* zeros, which was denoted mply by $P(\tau)$ in (I).

The following identity between $p(n, \tau)$ and $P_n(\tau)$ is roved in Appendix I. If $n \geq 2$,

$$p''(n, \tau) = \beta[P_n(\tau) - 2P_{n-1}(\tau) + P_{n-2}(\tau)], \tag{3}$$

here β is the expected number of zeros per unit time. or $n = 0$ and 1,

$$p''(0, \tau) = \beta P_0(\tau); \tag{4}$$

$$p''(1, \tau) = \beta[P_1(\tau) - 2P_0(\tau)]. \tag{5}$$

q. (4) has been given previously by Kohlenberg.[3]

Now if (2) is differentiated twice with respect to τ provided the second derivatives exist), and (3)–(5) are ubstituted, the following series is obtained in $P_n(\tau)$:

$$\frac{r''(\tau)}{4\beta} = \sum_{n=0}^{\infty} (-1)^n P_n(\tau). \tag{6}$$

If $P_n(\tau)$ is neglected when τ is small and $n \geq 1$, then $P_0(\tau) \cong r''(\tau)/4\beta$. This approximation agrees with 25) in (I). Special cases, in which this approximation not valid, will be treated later.

Another infinite series involving $P_n(\tau)$ may be derived lso. Let $U(\tau)d\tau$ be the conditional probability that a zero ccurs between $t + \tau$ and $t + \tau + d\tau$, given a zero at time t. For Gaussian noise, $U(\tau)d\tau$ has been derived by Rice.[4] The infinite series is

$$U(\tau) = \sum_{n=0}^{\infty} P_n(\tau). \tag{7}$$

The interpretation is relatively simple. If there is a zero in $(t + \tau, t + \tau + d\tau)$, it must be the first, or the second, or in general, the $(n + 1)$st zero after the one at t, where n is some nonnegative integer.

If $P_n(\tau)$ is neglected when τ is small and $n \geq 1$, then $P_0(\tau) \cong U(\tau)$. A better approximation than this one [and better than that obtained from the first term of (6)] can be found by adding (6) and (7). Then

$$\frac{1}{2}\left[U(\tau) + \frac{r''(\tau)}{4\beta}\right] = \sum_{m=0}^{\infty} P_{2m}(\tau) = Q(\tau). \tag{8}$$

$Q(\tau)d\tau$ is the conditional probability[5] that a downward crossing occurs between $t + \tau$ and $t + \tau + d\tau$, given an upward crossing at t. Rice has suggested $Q(\tau)$ as an approximation to $P_0(\tau)$ when τ is not too large. For a given value of τ, $Q(\tau)$ is obviously a better approximation than either $U(\tau)$ or $r''(\tau)/4\beta$, since in (8) we need not neglect $P_1(\tau)$ but only $P_2(\tau)$, $P_4(\tau)$, etc.

Still another series results if (6) is subtracted from (7). Then

$$\frac{1}{2}\left[U(\tau) - \frac{r''(\tau)}{4\beta}\right] = \sum_{m=0}^{\infty} P_{2m+1}(\tau). \tag{9}$$

For small τ the left member provides an approximation to $P_1(\tau)$.

The foregoing series in $P_n(\tau)$ is expressed in terms of $r''(\tau)$, β, and $U(\tau)$. $r''(\tau)$ is the second derivative of the autocorrelation function of $x(t)$, the clipped waveform. $r''(\tau)$ is also proportional to the autocorrelation of $x'(t)$, the waveform after infinite clipping and differentiation. Furthermore, by (12) of (I), β is equal to $-\frac{1}{2}r'(0+)$.

The function $U(\tau)$ can be related to another statistical property of the clipped signal, namely, the fourth moment,

$$w(\tau_1, \tau_2, \tau_3) = E[x(t)x(t + \tau_1)x(t + \tau_2)x(t + \tau_3)], \tag{10}$$

by means of the relation

$$U(\tau) = \frac{1}{4\beta}\left(\frac{\partial^2 w}{\partial \tau_1 \, \partial \tau_3}\right)_{0+,\tau,\tau+}. \tag{11}$$

This result is derived in Appendix II.

Initial Behavior of $P_0(\tau)$

It was shown in (I) that when $r(\tau)$ is nearly linear for small τ, the initial behavior of the probability density is $P_0(\tau) \cong r''(\tau)/4\beta$, and that this approximation becomes better as $\tau \to 0$. This section considers the initial value $P_0(0)$, particularly when $r''(0+) \neq 0$.

[2] S. O. Rice, "Mathematical analysis of random noise," *Bell ys. Tech. J.*, vol. 23, pp. 282–332; July, 1944, and vol. 24, pp. 46–56; January, 1945, see (2.7–2).

[3] A. Kohlenberg, "Notes on the Zero Distribution of Gaussian oise," M. I. T., Lincoln Lab., Lexington, Mass., Tech. Memo. , p. 4; October, 1953.

[4] Rice, *op. cit.*, (3.4–10).

[5] This function is discussed in (I). For Gaussian noise, $Q(\tau)d\tau$ is given by Rice, *op. cit.*, (3.4–1).

Consider the probability $p(n, \tau)$ that an interval $(t, t + \tau)$ contains exactly n zeros. By normalization,

$$\sum_{n=0}^{\infty} p(n, \tau) = 1. \tag{12}$$

Furthermore, the expected number of zeros in $(t, t + \tau)$ is $\beta\tau$; therefore

$$\sum_{n=1}^{\infty} np(n, \tau) = \beta\tau. \tag{13}$$

A third equation in $p(n, \tau)$ is the series (2) for the autocorrelation function $r(\tau)$.

Still another series involving these functions $p(n, \tau)$ can be written for the quantity $2\beta U(\tau)$. If $n(\tau)$ is the number of zeros observed in an interval $(t, t + \tau)$, then the mean square is

$$E[n^2(\tau)] = \sum_{n=1}^{\infty} n^2 p(n, \tau). \tag{14}$$

If we differentiate twice with respect to τ and $p''(n, \tau)$ is eliminated by means of (3) and (5), then it is found that

$$E'' [n^2(\tau)] = 2\beta \sum_{n=0}^{\infty} P_n(\tau), \tag{15}$$

or by (7),

$$E'' [n^2(\tau)] = 2\beta U(\tau). \tag{16}$$

For Gaussian noise, this result agrees with the equations of Steinberg *et al.*,[6] who expressed $E[n^2(\tau)]$ in terms of a single definite integral.

If (12), (13), and (2) are differentiated twice with respect to τ, and (16) is rewritten, we have the following four equations in $p''(n, \tau)$:

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} p''(n, \tau), \\ 0 &= \sum_{n=1}^{\infty} np''(n, \tau), \\ r''(\tau) &= \sum_{n=0}^{\infty} (-1)^n p''(n, \tau), \\ 2\beta U(\tau) &= \sum_{n=1} n^2 p''(n, \tau). \end{aligned} \tag{17}$$

Suppose that the functions $p(n, \tau)$ are regular functions of τ when $\tau \geq 0$. Then they may be expanded in a series about $\tau = 0$,

$$p(n, \tau) = p(n, 0) + p'(n, 0)\tau + p''(n, 0)\frac{\tau^2}{2} + \cdots .$$

In the case of $p(0, \tau)$ the leading term is unity, since it is certain that an interval of zero length contains no zeros. For $p(1, \tau)$ the leading term is $\beta\tau$, for $p(1, \tau) \to \beta d\tau$ as $\tau \to d\tau$. The leading term of $p(2, \tau)$ is of order $0\ (\tau^2)$, since $p(2, \tau)$ must become negligible compared to $p(1, \tau)$ as $\tau \to d\tau$.

Now suppose that the leading term of $p(3, \tau)$ is of order $0\ (\tau^3)$. More specifically, assume that $p''(n, 0) = 0$ when $n \geq 3$. Then when $\tau = 0$ in (17), we have four equations in three unknowns, $p''(0, 0)$, $p''(1, 0)$, and $p''(2, 0)$. These equations do not possess a consistent solution unless $U(0) = r''(0)/4\beta$. As will be seen, this restriction is not always satisfied.

If the conditions are relaxed slightly, and it is assumed that $p''(n, 0) = 0$ when $n \geq 4$, then when $\tau = 0$ in (17) we have four equations in four unknowns. The solutions are

$$\begin{aligned} p''(0, 0) &= \tfrac{1}{2}\beta U(0) + \tfrac{1}{8}r''(0); \\ p''(1, 0) &= -\tfrac{1}{2}\beta U(0) - \tfrac{3}{8}r''(0); \\ p''(2, 0) &= -\tfrac{1}{2}\beta U(0) + \tfrac{3}{8}r''(0); \\ p''(3, 0) &= \tfrac{1}{2}\beta U(0) - \tfrac{1}{8}r''(0). \end{aligned} \tag{18}$$

The initial values of the probability densities $P_n(\tau)$ may be obtained by using (81) of Appendix I.

$$\begin{aligned} P_0(0) &= \frac{1}{2}\left[U(0) + \frac{r''(0)}{4\beta}\right]; \\ P_1(0) &= \frac{1}{2}\left[U(0) - \frac{r''(0)}{4\beta}\right]; \\ P_n(0) &= 0, \qquad n \geq 2. \end{aligned} \tag{19}$$

Thus, under the given assumptions on $p''(n, 0)$, Rice's approximation $Q(\tau)$ for $P_0(\tau)$ becomes exact as $\tau \to 0$. If $U(0) = r''(0)/4\beta$, then $P_0(0) = U(0) = r''(0)/4\beta$ and $P_1(0) = 0$.

Specialize now to the Gaussian process. If $\xi(t)$ is Gaussian and the normalized autocorrelation function is $\rho(\tau)$, then Rice[4] has shown that

$$U = \frac{2}{\pi}\left(\frac{1}{H} + \tan^{-1} H\right)\frac{r''}{4\beta}, \tag{20}$$

where

$$\begin{aligned} r'' &= (2/\pi)M_{23}(1 - \rho^2)^{-3/2}, \\ H &= M_{23}[M_{22}^2 - M_{23}^2]^{-1/2}, \\ M_{22} &= -\rho''(0)(1 - \rho^2) - \rho'^2, \\ M_{23} &= \rho''(1 - \rho^2) + \rho\rho'^2, \\ \beta &= [-\rho''(0)]^{1/2}/\pi, \end{aligned} \tag{21}$$

the dependence on τ being understood.

The autocorrelation function $\rho(\tau)$ is symmetric with respect to $\tau = 0$. If it is regular at $\tau = 0$, then it can be expanded in a power series containing only even powers,

$$\rho(\tau) = 1 - a\tau^2 + c\tau^4 + \cdots . \tag{22}$$

Using Rice's formulas, it is found that $r''(0) = 0$ and

$$\frac{2}{\pi}\left(\frac{1}{H} + \tan^{-1} H\right) \to 1$$

[6] H. Steinberg, P. M. Schultheiss, C. A. Wogrin, and F. Zweig, "Short-time frequency measurement of narrow-band random signals by means of a zero-counting process," *J. Appl. Phys.*, vol. 26, pp. 195–201; February, 1955.

s $\tau \to 0$. Then $U(0) = 0$, and by (19), $P_0(0) = 0$ and $P_1(0) = 0$. The initial behavior of $P_0(\tau)$ and $P_1(\tau)$ in his (regular) case has been discussed more fully by 'almer.[7]

Suppose, however, that $\rho(\tau)$ is not regular. If at $\tau = 0$ ll the derivatives of $\rho(\tau)$ exist up through the fourth, hen the previous analysis will still apply. In terms of the ower spectrum $W(f)$, where f is the frequency, it may be oncluded that $P_0(0) = 0$ if the integral

$$\int_0^\infty W(f)f^4\,df$$

xists. This result follows from the Wiener-Khintchine heorem.[8]

Suppose this integral does not exist. If $\rho''(0)$ does not xist either, then β is infinite, as in the case when $\xi(t)$ is he output from an RC low-pass filter and the input is vhite noise.[9] If $W(f)$ behaves asymptotically like f^{-4} s $f \to \infty$, then $\rho''(0)$ exists, but $\rho'''(\tau)$ is discontinuous t $\tau = 0$. $\rho(\tau)$ may be expanded in the series,

$$\rho(\tau) = 1 - a\tau^2 + b\,|\tau|^3 + \cdots . \tag{23}$$

n this case $r''(0) > 0$ and

$$\lim_{\tau \to 0} \frac{2}{\pi}\left(\frac{1}{H} + \tan^{-1} H\right) = \frac{2\sqrt{3}}{\pi} + \frac{1}{3}. \tag{24}$$

'hen $U(0) > r''(0)/4\beta$ and, by (19), $P_0(0) > 0$ and $P_1(0) > 0$. It was shown by Palmer[7] that, when $\xi(t)$ is aussian and $\rho(\tau)$ is of the form (22), successive axis-rossing intervals must be statistically dependent. Palmer howed that as $\tau \to 0$, $P_0(\tau)$ becomes asymptotically roportional to τ and $P_1(\tau)$ behaves like τ^4, whereas if uccessive intervals were independent, then $P_1(\tau)$ would e a convolution of $P_0(\tau)$ with itself and therefore $P_1(\tau)$ vould behave initially like τ^3.

A similar analysis applies in the case of (23). It is seen hat $P_0(0) > 0$ and $P_1(0) > 0$, whereas if successive inter-als were independent the convolution integral would yield $P_1(\tau)$ which began at zero and behaved linearly for mall τ.

The dependence between successive axis-crossing tervals is discussed more fully in the following sections.

Approximate Solutions for $P_0(\tau)$: Statistically Independent Intervals

This section discusses methods for determining the robability density $P_0(\tau)$ and its moments under the ssumption that successive axis-crossing intervals are tatistically independent.

Extensive use will be made of the Laplace transform $y(s)$ of $Y(\tau)$,

$$y(s) = L\{Y(\tau)\} = \int_0^\infty e^{-s\tau}Y(\tau)\,d\tau. \tag{25}$$

The following Laplace transforms are defined:

$$L\{r''(\tau)/4\beta\} = g(s); \qquad L\{U(\tau)\} = u(s); \qquad L\{P_n(\tau)\} = p_n(s). \tag{26}$$

Transform the infinite series (6) and (7).

$$g(s) = \sum_{n=0}^{\infty} (-1)^n p_n(s) \tag{27}$$

$$u(s) = \sum_{n=0}^{\infty} p_n(s). \tag{28}$$

If it is presumed that successive axis-crossing intervals are statistically independent, then the probability densities of sums of intervals are given by convolutions of $P_0(\tau)$, or in terms of the transforms,

$$p_n(s) = p_0^{n+1}(s), \tag{29}$$

and (27) and (28) yield solutions[10] for $p_0(s)$. Respectively,

$$p_0(s) = g(s)/[1 - g(s)]; \tag{30}$$

$$p_0(s) = u(s)/[1 + u(s)]. \tag{31}$$

These two solutions cannot be expected to be identical if the assumption of independence is incorrect.

Equations related to (30) have been given by Lawson and Uhlenbeck[11] and also by Huggins.[12] The three random square-wave models of $x(t)$ given in (I) (*i.e.*, all but the case of clipped Gaussian noise) can be solved exactly by this equation.

Eq. (31) is a basic equation in renewal theory.[10] Miller and Freund[13] have suggested its application to Gaussian noise and have solved it by assuming a simple function for $U(\tau)$. They did not test their assumption (independence) by a comparison with experiment.

It is clear from the definition (25) that the moments of $P_0(\tau)$ may be obtained from derivatives of the transform $p_0(s)$, evaluated at $s = 0$. The easiest method is to expand the functions in power series,

$$p_0(s) = 1 - E(\tau)s + E(\tau^2)s^2/2 - \cdots , \qquad g(s) = g(0) + g'(0)s + g''(0)s^2/2 + \cdots , \tag{32}$$

where E denotes the expected value, and substitute into (30). If $r(\tau)$ and its derivatives vanish at infinity, then $g(0) = -r'(0+)/4\beta$. By (12) in (I), $r'(0+) = -2\beta$;

[7] D. S. Palmer, "Properties of random functions," *Proc. Cam-ridge Phil. Soc.*, vol. 52, pp. 672–686; October, 1956. See pp. 679–80.

[8] Rice, *op. cit.*, (2.1–6).

[9] In that case $\xi(t)$ is a Markoff process. Although β and $P_0(\tau)$ do ot exist, there are other probabilities which can be calculated. See J. F. Siegert, "On the Roots of Markoffian Random Functions," he RAND Corp., Santa Monica, Calif., Rep. RM-447; September, 950.

[10] Cf. M. S. Bartlett, "An Introduction to Stochastic Processes," Cambridge University Press, Cambridge, Eng., p. 59; 1955. Also, W. Feller, "Probability Theory and its Applications," John Wiley and Sons, Inc., New York, N. Y., ch. 12 and 13; 1950.

[11] J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," McGraw-Hill Book Co., Inc., New York, N. Y., p. 45, (36b); 1950.

[12] W. H. Huggins, "Signal-flow graphs and random signals," Proc. IRE, vol. 45, pp. 74–86; January, 1957. See (48).

[13] I. Miller and J. E. Freund, "Some Distribution Theory Connected with Gaussian Processes," Virginia Polytech. Inst., Blacksburg, Va., Dept. of Statist. Tech. Rep. 19, p. 129; May, 1956.

therefore, $g(0) = \frac{1}{2}$. Also $g'(0) = -1/4\beta$ and $g''(0) = A/2\beta$, where

$$A = \int_0^\infty r(\tau)\,d\tau. \tag{33}$$

The last two results, $g'(0)$ and $g''(0)$, require integration by parts. It is assumed that $\tau r(\tau) \to 0$ and $\tau^2 r'(\tau) \to 0$ as $\tau \to \infty$.

Using the above series in (30), it is found, by matching coefficients of equal powers of s, that $E(\tau) = 1/\beta$ [in agreement with the discussion in (I)] and that

$$E(\tau^2) = 2A/\beta + 1/\beta^2. \tag{34}$$

Then the variance is $E(\tau^2) - [E(\tau)]^2$, or

$$\sigma^2(\tau) = 2A/\beta. \tag{35}$$

By the Wiener-Khintchine theorem,[8] A is proportional to the spectral density (after clipping) at zero frequency. The spectrum of clipped *Gaussian* noise has been discussed by Van Vleck.[14]

The renewal equation (31) may also be used for the computation of moments, but with some difficulty. The function $u(s)$ does not exist at $s = 0$, since $U(\tau)$ does not vanish at infinity. As $\tau \to \infty$, $U(\tau)d\tau \to \beta d\tau$ since the probability of a zero in $(t + \tau, t + \tau + d\tau)$, given a zero at t, becomes independent of τ. If β is subtracted and a new variable is defined,

$$V(\tau) = U(\tau) - \beta, \tag{36}$$

with a Laplace transform

$$L\{V(\tau)\} = v(s) = u(s) - \beta/s, \tag{37}$$

$u(s)$ may be eliminated from (31) and $v(s)$ expanded about $s = 0$. The results are that the mean is again $E(\tau) = 1/\beta$ and that the variance is

$$\sigma^2(\tau) = (1 + 2B)/\beta^2, \tag{38}$$

where

$$B = \int_0^\infty [U(\tau) - \beta]\,d\tau. \tag{39}$$

It cannot be expected that the two results (35) and (38) will be equal if the assumption of independent intervals is incorrect.

The function $V(\tau)$ is proportional to what Bartlett[15] calls the "covariance density" of a stationary point process, when the points are the zeros of $\xi(t)$.

For Gaussian noise, the solution for the probability density $P_0(\tau)$ is not feasible by the Laplace transform method, since $r''(\tau)$ and $U(\tau)$ are too complicated. Instead of (30) and (31), the two corresponding Volterra integral equations may be written,

$$P_0(\tau) = r''(\tau)/4\beta + [r''(\tau)/4\beta] * P_0(\tau), \tag{40}$$

$$P_0(\tau) = U(\tau) - U(\tau) * P_0(\tau), \tag{41}$$

[14] Lawson and Uhlenbeck, *op. cit.*, p. 57.
[15] Bartlett, *op. cit.*, p. 166.

where (*) denotes the convolution process

$$Y_1(\tau) * Y_2(\tau) = \int_0^\tau Y_1(l)Y_2(\tau - l)\,dl. \tag{42}$$

These integral equations can be solved numerically, and, as stated before, the results generally will be different between (40) and (41). Palmer[16] has solved (41) when $\rho(\tau) = \exp(-\frac{1}{2}\tau^2)$.

Quasi-Independent Intervals

A new integral equation for $P_0(\tau)$ is derived now. The equation is quite similar to (40) and (41), but the restrictive assumption is somewhat weaker.

Apply the Laplace transformation to the infinite series (8) and (9). Then

$$\tfrac{1}{2}[u(s) + g(s)] = \sum_{m=0}^{\infty} p_{2m}(s); \tag{43}$$

$$\tfrac{1}{2}[u(s) - g(s)] = \sum_{m=0}^{\infty} p_{2m+1}(s). \tag{44}$$

Instead of assuming that all successive axis-crossing intervals are independent, now assume that a given interval is independent of the sum of the $(2m + 2)$ intervals immediately preceding. This statement applies for *all* values of m, where $m = 0, 1, 2, \cdots$. This assumption will be called *quasi-independence.* Now $P_{2m+1}(\tau)$ is the probability density of the sum of $(2m + 2)$ successive intervals. By our assumption,

$$p_{2m+1}(s)p_0(s) = p_{2m+2}(s). \tag{45}$$

If both sides of (44) are multiplied by $p_0(s)$ and the relation (45) is used, then $p_2(s), p_4(s), p_6(s), \cdots$, may be eliminated by the use of (43). Then solve for $p_0(s)$.

$$p_0(s) = \frac{u(s) + g(s)}{2 + u(s) - g(s)}. \tag{46}$$

Under this solution the mean interval is again $E(\tau) = 1/\beta$ and the variance $\sigma^2(\tau)$ is the same as (38). In general, however, the solution $P_0(\tau)$ will be different from the solution of either (40) or (41).

As with (40) and (41), the best method for a practical solution is to write down the corresponding integral equation,

$$P_0(\tau) = \frac{1}{2}\left[U(\tau) + \frac{r''(\tau)}{4\beta}\right] - \frac{1}{2}\left[U(\tau) - \frac{r''(\tau)}{4\beta}\right] * P_0(\tau), \tag{47}$$

and to solve it numerically. Notice that the first term in the right member is $Q(\tau)$, which Rice has given as an approximation to $P_0(\tau)$ when τ is not too large. The solution of (47) provides a correction to Rice's approximation.

For Gaussian noise, (47) has been solved numerically for several different power spectra. The normalized

[16] Palmer, *op. cit.*, p. 678.

utocorrelation function was obtained by the Wiener-Khintchine theorem; then $U(\tau)$ and $r''(\tau)/4\beta$ were btained from (20) and (21). The convolution integral as approximated by the trapezoidal rule, using a mesh idth of $\Delta\tau$. Starting with $P_0(0)$ given by (19), we solved uccessively for $P_0(\Delta\tau)$, $P_0(2\Delta\tau)$, etc.[17] The most time-onsuming part of the calculation is the computation of $U(\tau)$ and not the solution of the integral equation.

The following spectra have been considered.

Case 1:

$$W(f) = \frac{(2\pi f)^{2m}}{(1 + 4\pi^2 f^2)^n}, \tag{48}$$

here $m = 0$, $n = 2$. In this case,

$$\rho(\tau) = (1 + |\tau|)e^{-|\tau|}. \tag{49}$$

Case 2: $m = 0$, $n = 3$, and correspondingly

$$\rho(\tau) = (1 + |\tau| + \tfrac{1}{3}\tau^2)e^{-|\tau|}. \tag{50}$$

Case 3: $m = 2$, $n = 4$, and correspondingly

$$\rho(\tau) = (1 + |\tau| - 2\tau^2 + \tfrac{1}{3}|\tau|^3)e^{-|\tau|}. \tag{51}$$

Case 4 is the spectrum of the output of a particular ow-pass Butterworth filter, if the input is white noise. et

$$W(f) = [1 + (f/f_0)^{14}]^{-1}. \tag{52}$$

hen, if $2\pi f_0 = 1$,

$$(\tau) = \sin\frac{\pi}{14}\left\{e^{-|\tau|} + 2\sum_{n=1}^{3} e^{-|\tau|\cos(n\pi/7)}\cos\left[\frac{n\pi}{7} - |\tau|\sin\frac{n\pi}{7}\right]\right\}. \tag{53}$$

Case 5 is the ideal low-pass spectrum.

$$\begin{aligned} W(f) &= \text{constant}, \quad 0 < f < f_0, \\ &= 0, \quad f \geq f_0. \end{aligned} \tag{54}$$

f $2\pi f_0 = 1$, then the normalized autocorrelation function s

$$\rho(\tau) = (\sin\tau)/\tau. \tag{55}$$

Case 6 is the ideal octave band-pass spectrum.

$$\begin{aligned} W(f) &= 0, \quad 0 < f \leq f_0, \\ &= \text{constant}, \quad f_0 < f < 2f_0, \\ &= 0, \quad f \geq 2f_0. \end{aligned} \tag{56}$$

or simplicity, again let $2\pi f_0 = 1$. Then

$$\rho(\tau) = (\sin 2\tau - \sin\tau)/\tau. \tag{57}$$

Cases 1 and 3 belong to the class where $W(f)$ behaves ke f^{-4} as $f \to \infty$; therefore, the limit (24) must be used to determine $U(0)$. In the other cases, $U(0) = r''(0)/4\beta = 0$.

The solutions of (47) under these six conditions were obtained with the aid of an IBM 650 Magnetic Drum Calculator and are shown in Fig. 1. The first four are compared with the experimental results of Favreau, Low, and Pfeffer.[18] The actual experimental points are shown when available. In Fig. 1(b) and 1(d), the agreement is very good. In Fig. 1(a) and 1(c), the agreement is very good also except near $\tau = 0$. These discrepancies may be caused by the absence of extremely high frequencies in the experimental measurement; *i.e.*, the actual spectrum may have deviated from the theoretical function $W(f)$. In some cases, the solution $P_0(\tau)$ dips negative and therefore cannot be accepted for large intervals τ. Evidently the assumption of quasi-independence is not valid here.

In each part of Fig. 1 Rice's function $Q(\tau)$ also has been plotted. In those cases in which the upper frequency cutoff is extremely sharp (cases 5 and 6), it is found that $Q(\tau)$ and the solution of (47) agree closely over most of the range of axis-crossing intervals. When they agree, it may be concluded that $Q(\tau)$ is a good approximation to the *true* probability density $P_0(\tau)$. Beyond the range of agreement, the solution of (47) soon dips negative; therefore, the range of validity of the solution of (47) cannot be much greater than the range of applicability of Rice's function $Q(\tau)$. An extreme example is the ideal octave-band case 6. On the other hand, when the cutoff is gradual (cases 1–3), then $Q(\tau)$ is not at all useful as an approximation to the true $P_0(\tau)$, but the solution of (47) is quite accurate.

What can be concluded about the assumption leading to (47), *i.e.*, the assumption of quasi-independence? On the basis of these examples it appears that when the high-frequency cutoff is gradual, the assumption is good. When the cutoff is extremely sharp, the assumption is poor. In any case the assumption is not good for extremely large intervals τ therefore; it should not be used for the calculation of moments.

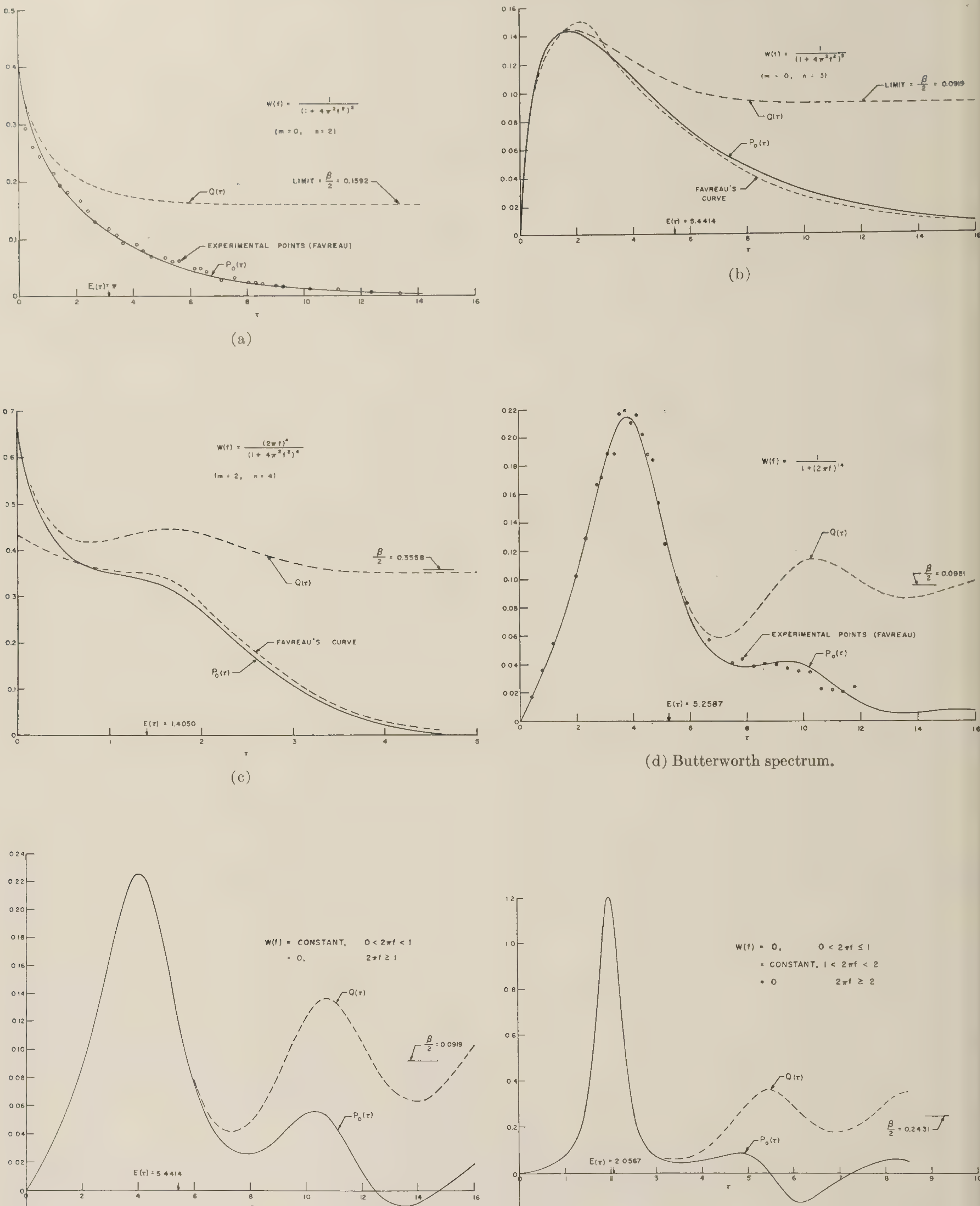
The assumption of *independence* also has been tested by the solution (not shown in Fig. 1) of (40) and (41). In case 2 the two results are extremely close to each other and to the curves in Fig. 1(b). Thus it appears that renewal theory can be used, at least approximately, for this type of spectrum. In all the other cases, however, the results are badly inconsistent and the assumption of independence should be rejected.

The question of dependence is discussed again in the following section.

Favreau, Low, and Pfeffer[18] have suggested that for case 1 $P_0(\tau)$ may be fitted exactly by an exponential curve. Since $E(\tau) = \pi$, the initial value would be $P_0(0) = 1/\pi$, in conflict with (19), from which $P_0(0) = 0.40600$.

[17] A more refined treatment is considered by L. Fox and E. T. oodwin, "The numerical solution of non-singular linear integral quations," *Phil. Trans. Roy. Soc. London*, vol. A245, pp. 501–534; ebruary, 1953. See p. 522.

[18] R. R. Favreau, H. Low, and I. Pfeffer, "Evaluation of complex statistical functions by an analog computer," 1956 IRE National Convention Record, pt. 4, pp. 31–37.



(e) Ideal low-pass spectrum.

(f) Ideal octave-band spectra.

Fig. 1—The probability density $P_0(\tau)$ of axis-crossing intervals for six examples of Gaussian noise, under the assumption of quasi-independence, obtained by the solution of the integral equation (47). Also shown are Rice's function $Q(\tau)$ and its asymptotic limit, and the theoretical value of the mean interval, $E(\tau) = 1/\beta$. The experimental results are taken from the work of Favreau, Low, and Pfeffer.[18]

Thus, if the hypothesis is valid [$p''(n, 0) = 0$ when $n \geq 4$], the solution for $P_0(\tau)$ cannot be exactly exponential.

Markoff Chain of Intervals

This section shows the derivation of the variance of axis-crossing intervals[19] and the correlation coefficient between two successive intervals, under the assumption that the successive intervals form a Markoff chain in the wide sense.

The basic equations will again be (27) and (28), which involve the infinite series in the transforms $p_n(s)$. The first step is to express $p_n(s)$ as a function of $p_0(s)$. Let

$$p_n(s) = a_n(s)p_0^{n+1}(s), \tag{58}$$

where $a_n(s)$ has not yet been determined. This equation is a generalization of (29), which followed when successive intervals were independent. This relation will be substituted into (27) and (28) and the series summed in n. Then the second moment in the expansion of $p_0(s)$ will be identified, as in (32). But first, $a_n(s)$ must be determined.

If $p_0(s)$ and $p_n(s)$ are expanded in powers of s, specifying the appropriate moments, then (58) will yield a series for $a_n(s)$. Now $P_n(\tau)$ is the probability density of the sum of $(n + 1)$ successive intervals, $\tau_1, \tau_2, \cdots, \tau_{n+1}$, and the mean value of the sum is $(n + 1)/\beta$. Then from (58),

$$\begin{aligned} a_n(s) &= \frac{1 - \dfrac{n+1}{\beta}s + \dfrac{s^2}{2}E\left[\left(\sum_{i=1}^{n+1}\tau_i\right)^2\right] - \cdots}{\left[1 - \dfrac{s}{\beta} + \dfrac{s^2}{2}E(\tau^2) - \cdots\right]^{n+1}} \\ &= 1 + \left\{E\left[\left(\sum_{i=1}^{n+1}\tau_i\right)^2\right] - \frac{n(n+1)}{\beta^2} \right. \\ &\qquad \left. - (n+1)E(\tau^2)\right\}\frac{s^2}{2} + O(s^3). \end{aligned} \tag{59}$$

Eq. (59) can be simplified considerably if the mean square of the sum of intervals can be expressed in terms of the variance of a single interval. Use will be made of the correlation coefficient κ_{ij} between the ith and jth axis-crossing intervals. Let

$$\kappa_{ij} = [E(\tau_i\tau_j) - 1/\beta^2]/\sigma^2, \qquad (i \neq j), \tag{60}$$

where $\sigma^2 = \sigma^2(\tau_i) = \sigma^2(\tau_j)$, or simply $\sigma^2(\tau)$. $\sigma^2(\tau)$ is not necessarily given by (35) or (38), since we have not assumed independent intervals.

Now the assumption is made that the successive axis-crossing intervals form a Markoff chain in the wide sense,[20] or that

$$\kappa_{ij} = \kappa^{|i-j|}, \tag{61}$$

where κ is the correlation coefficient between two successive intervals. Under this assumption, the mean square of the sum of n successive intervals is

$$E\left[\left(\sum_{i=1}^{n}\tau_i\right)^2\right] = \frac{n^2}{\beta^2} + \frac{\sigma^2}{(1-\kappa)^2}[-2\kappa + n(1-\kappa^2) + 2\kappa^{n+1}]. \tag{62}$$

n may be replaced by $(n + 1)$ and the result substituted into (59). For $a_n(s)$,

$$a_n(s) = 1 + \frac{\kappa[n - (n+1)\kappa + \kappa^{n+1}]}{(1-\kappa)^2}\sigma^2 s^2 + O(s^3). \tag{63}$$

If $\kappa = 0$ there is no dependence between successive intervals except possibly in the third and higher moments.

Eq. (58) can be substituted now into the infinite series (27) and (28). In the case of (27), the series is summed in n, the functions $p_0(s)$ and $g(s)$ expanded as in (32), and finally, the coefficients of s^2 identified. Solving for σ^2,

$$\sigma^2(\tau) = \frac{2A}{\beta}\frac{1+\kappa}{1-\kappa}. \tag{64}$$

If $\kappa = 0$ we have the previous result (35).

If (58) is substituted into the other infinite series (28), a singularity is found. However, if $u(s)$ is eliminated by means of (37), the series summed in n, and the functions $p_0(s)$ and $v(s)$ expanded about $s = 0$, all singularities disappear. The coefficients of s^2 may be matched again and solved for σ^2. Then

$$\sigma^2(\tau) = \frac{1+2B}{\beta^2}\frac{1-\kappa}{1+\kappa}. \tag{65}$$

If $\kappa = 0$ we have the previous result (38).

Now if the Markoff assumption is valid, the two results (64) and (65) may be equated and solved for κ and $\sigma^2(\tau)$. It is found that $\sigma^2(\tau)$ is the geometric mean of the two previous results (35) and (38),

$$\sigma^2(\tau) = \left[\frac{2A}{\beta}\frac{1+2B}{\beta^2}\right]^{1/2}, \tag{66}$$

and that the correlation coefficient κ may be obtained from the ratio of these same results by

$$\frac{1-\kappa}{1+\kappa} = \left(\frac{2A}{\beta}\right)^{1/2} \bigg/ \left(\frac{1+2B}{\beta^2}\right)^{1/2}. \tag{67}$$

In the last equation, the positive root was chosen so that $|\kappa| < 1$.

If the two results (35) and (38) (based on the assumption of independent intervals) are equal, then (67) gives the result $\kappa = 0$.

$\sigma(\tau)$ and κ have been computed for several examples of Gaussian noise. The integrals A and B were obtained numerically by Simpson's rule on an IBM 650. For Gaussian noise, $r(\tau)$ is given by (9) in (I):

$$r(\tau) = \frac{2}{\pi}\sin^{-1}\rho(\tau). \tag{68}$$

The results are given in Table I, together with the mean value $E(\tau)$.

[19] This result was first presented to the Institute of Mathematical Statistics in Washington, D. C., March 7, 1957. See J. A. McFadden, "The variance of zero-crossing intervals," *Ann. Math. Statist.*, vol. 28, p. 529; June, 1957.

[20] J. L. Doob, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y., p. 233, (8.2); 1953.

TABLE I

MEAN AXIS-CROSSING INTERVAL $E(\tau)$, STANDARD DEVIATION $\sigma(\tau)$, AND CORRELATION COEFFICIENT κ BETWEEN TWO SUCCESSIVE INTERVALS FOR GAUSSIAN NOISE OF VARIOUS SPECTRA

Case	Spectrum $W(f)$ (eq. no.)	Correlation $\rho(\tau)$ (eq. no.)	$E(\tau)=\frac{1}{\beta}$	$\sigma(\tau)$	κ
1	(48), $m=0$, $n=2$	(49)	3.1416	3.22	0.058
2	(48), $m=0$, $n=3$	(50)	5.4414	4.68	0.005
3	(48), $m=2$, $n=4$	(51)	1.4050	0.775	0.486
5	(54) ideal low-pass	(55)	5.4414	3.74	0.009
6	(56) ideal octave band	(57)	2.0567	0.565	0.442

As might be expected, the ratio $\sigma(\tau)/E(\tau)$ is smallest in the two cases 3 and 6 where the bandwidth is relatively narrow, *i.e.*, comparable to the midband frequency. These are also the cases in which successive axis-crossing intervals are highly correlated. In the low-pass cases 1, 2, and 5, the correlation κ is small.

What can be said about the *true* value of the correlation coefficient between two successive intervals, without restoring to the Markoff hypothesis? Suppose it is assumed that $\kappa = 0$. Then the expressions for variance (64) and (65) degenerate to (35) and (38), respectively. If the numerical results of these two formulas agree closely, then the assumption $\kappa = 0$ cannot be far wrong. The true value of the correlation coefficient must be quite small, although not necessarily equal to the value of κ obtained from (67).

On the other hand, if the numerical results of (35) and (38) differ widely, then the assumption $\kappa = 0$ must be rejected. The true correlation must be large, although the actual value may differ considerably from κ [obtained from (67)], depending on the suitability of the Markoff assumption.

The ideal low-pass spectrum (54) is particularly significant. It was shown by a numerical solution of (40) that the assumption of independent intervals is far from correct, since the solution $P_0(\tau)$ dips far below the axis, much more than in Fig. 1(e). Yet the correlation κ is nearly zero. Evidently the dependence between successive intervals is contained largely in the moments higher than the second. On the other hand, for a low-pass spectrum such as (48) ($m = 0$, $n = 3$), where the cutoff is gradual, it may be concluded not only that κ is nearly zero but also that successive intervals are nearly independent, since (by another calculation) the solutions of (40) and (41) appeared quite close to that of (47) and to the experimental result.

CONCLUSION

The results of this paper and the previous paper (I) may be summarized as follows.

1) For a given process $\xi(t)$, the analysis of the clipped signal $x(t)$ aids the understanding of the axis-crossing problem.

2) For Gaussian noise, the assumption of statistically independent intervals is usually not valid. The assumption of quasi-independence is a fairly good one when the high-frequency cutoff is not extremely sharp. This assumption should not be used to calculate moments.

3) The mean value $E(\tau) = 1/\beta$ can be obtained from (12) in (I), *i.e.*, $\beta = -\frac{1}{2}r'(0+)$. This formula reduces to Rice's result in the Gaussian case.

4) Under the assumption that successive intervals form a Markoff chain in the wide sense, the variance $\sigma^2(\tau)$ of the interval distribution can be found from (66), and the correlation coefficient κ between successive intervals can be found from (67). These calculations require the knowledge of the functions $r(\tau)$ and $U(\tau)$. For Gaussian noise these functions are known and the parameters $\sigma^2(\tau)$ and κ can be obtained by numerical integration. The Markoff assumption has not been tested by experiment.

5) In some examples of Gaussian noise, κ is very small but the statistical dependence between successive intervals is nevertheless strong.

It is felt that the best hope for future progress in this problem lies in the invention of simplified models for stochastic processes. These models would not conform precisely to actual noise signals but would permit analytical results. The models must not assume independence but must provide for dependence between successive intervals.

APPENDIX I

THE RELATION BETWEEN $p(n,\tau)$ AND $P_n(\tau)$

Now, (3) will be derived. This identity applies not only to axis-crossings but to more general stationary point processes.[15] It is *not* assumed that successive intervals are independent.

Let $p(n, \tau)$ be the probability that a given interval $(t, t + \tau)$ contains exactly n *points*, and let $P_n(\tau)$ be the probability density of the interval between the mth and $(m + n + 1)st$ points, where m is arbitrary. β is the expected number of points per unit time.

Consider the following events E_1, E_2, and E_3:

E_1: There are exactly n points in the interval $(t, t + \tau)$.

E_2: There are exactly n points in the interval $(t, t + \tau + d\tau)$.

E_3: There are exactly n points in $(t, t + \tau)$ and there is one point in the adjacent interval $(t + \tau, t + \tau + d\tau)$. The probability of more than one point in an infinitesimal interval is neglected.

E_2 can be divided into two alternatives:

$E_2^{(1)}$: There are exactly n points in $(t, t + \tau)$ and none in $(t + \tau, t + \tau + d\tau)$.

$E_2^{(2)}$: There are exactly $(n - 1)$ points in $(t, t + \tau)$ and there is one in $(t + \tau, t + \tau + d\tau)$.

By the conservation of total probability, we may write two identities concerning the probabilities of these events.

$$\Pr\{E_1\} = \Pr\{E_2^{(1)}\} + \Pr\{E_3\} \tag{69}$$

$$\Pr\{E_2\} = \Pr\{E_2^{(1)}\} + \Pr\{E_2^{(2)}\}. \tag{70}$$

ubtraction of these two equations yields the following:

$$\Pr\{E_2\} - \Pr\{E_1\} = \Pr\{E_2^{(2)}\} - \Pr\{E_3\}. \tag{71}$$

Now let these four probabilities be expressed. Clearly,

$$\Pr\{E_1\} = p(n, \tau); \tag{72}$$

$$\Pr\{E_2\} = p(n, \tau + d\tau); \tag{73}$$

nd the difference, except for higher-order terms of the 'aylor-series expansion, is

$$\Pr\{E_2\} - \Pr\{E_1\} = p'(n, \tau)\, d\tau. \tag{74}$$

The right member of (71) presents more difficulty. Let wo more events be defined:

E_4: There are not more than n points in $(t, t + \tau)$ and here is one in $(t + \tau, t + \tau + d\tau)$.

E_5: There are not more than $(n - 1)$ points in $(t, t + \tau)$ nd there is one in $(t + \tau, t + \tau + d\tau)$.

The desired probability $\Pr\{E_3\}$ is the difference

$$\Pr\{E_3\} = \Pr\{E_4\} - \Pr\{E_5\}. \tag{75}$$

ow

$$\Pr\{E_4\} = \beta\, d\tau \int_\tau^\infty P_n(l)\, dl, \tag{76}$$

nd

$$\Pr\{E_5\} = \beta\, d\tau \int_\tau^\infty P_{n-1}(l)\, dl. \tag{77}$$

hese equations may be interpreted as follows: $\beta d\tau$ is the robability of finding one point in $(t + \tau, t + \tau + d\tau)$. ooking backward in time,

$$\int_\tau^\infty P_n(l)\, dl$$

the conditional probability that the $(n + 1)$st point rior to the one in $(t + \tau, t + \tau + d\tau)$ occurred before the me t, given a point in $(t + \tau, t + \tau + d\tau)$. Then (76) the joint probability of finding one point in $(t + \tau,$ $+ \tau + d\tau)$ and not more than n points in $(t, t + \tau)$. q. (77) has a similar interpretation. Then by (75),

$$\Pr\{E_3\} = \beta\, d\tau \int_\tau^\infty [P_n(l) - P_{n-1}(l)]\, dl. \tag{78}$$

y similar reasoning,

$$\Pr\{E_2^{(2)}\} = \beta\, d\tau \int_\tau^\infty [P_{n-1}(l) - P_{n-2}(l)]\, dl. \tag{79}$$

hen if the probabilities (74), (78), and (79) are subituted into identity (71) and divided by $d\tau$, the result is

$$(n, \tau) = -\beta \int_\tau^\infty [P_n(l) - 2P_{n-1}(l) + P_{n-2}(l)]\, dl. \tag{80}$$

the second derivative $p''(n, \tau)$ exists, (80) may be fferentiated with respect to τ; (3) is the result.

The previous derivation applies only when $n \geq 2$; however, it may be extended easily to the cases $n = 0$, 1 provided that the interpretation $P_k(\tau) \equiv 0$, when $k < 0$, is made. The resulting equations are (4) and (5).

Eq. (3) may be considered as a difference equation in $P_n(\tau)$. Its solution is

$$P_n(\tau) = \frac{1}{\beta} \sum_{k=0}^{n} (n - k + 1) p''(k, \tau). \tag{81}$$

One simple example of the relationship between $p(n, \tau)$ and $P_n(\tau)$ is the following: if the number of points in an interval $(t, t + \tau)$ obeys the (discrete) Poisson distribution

$$p(n, \tau) = \frac{(\beta\tau)^n}{n!} e^{-\beta\tau}, \tag{82}$$

then the time interval between the mth and $(m + n + 1)$st points obeys the (continuous) gamma distribution

$$P_n(\tau) = \frac{\beta^{n+1}\tau^n}{n!} e^{-\beta\tau}. \tag{83}$$

Appendix II

The Relation between $U(\tau)$ and the Clipped Waveform

Now (11), which relates $U(\tau)$ to a property of the clipped waveform, is derived. Let $x_1 = x(t)$, $x_2 = x(t + \tau_1)$, $x_3 = x(t + \tau_2)$, $x_4 = x(t + \tau_3)$, where $0 < \tau_1 < \tau_2 < \tau_3$. The probability that $x_1 = -1$, $x_2 = +1$, $x_3 = +1$, and $x_4 = -1$, denoted by P_{-++-}, is

$$P_{-++-} = \frac{1}{16}[1 - r_{12} - r_{13} + r_{14} + r_{23} - r_{24} - r_{34} + w], \tag{84}$$

where $r_{ij} = E(x_i x_j)$ and $w = E(x_1 x_2 x_3 x_4)$. This equation is derived in the same manner as (14) in (I). The moments $E(1)$, $E(x_i)$, $E(x_i x_j)$, $E(x_i x_j x_k)$, and $E(x_1 x_2 x_3 x_4)$ are written in terms of the probabilities P_{++++}, P_{+++-}, etc. Then, set $E(1) = 1$ and $E(x_i) = 0$ and assume that $E(x_i x_j x_k) = 0$ $(i < j < k)$, and solve for P_{-++-}.[21] In the case of Gaussian noise, P_{-++-} is an example of the "quadrivariate normal integral," which has not been solved in closed form except for special values of the correlation matrix.[22]

The correlation coefficients in (84) may be replaced by the autocorrelation function of the appropriate time lags; *e.g.*, $r_{12} = r(\tau_1)$. The next step is to let $\tau_1 = dt$, $\tau_2 = \tau$, and $\tau_3 = \tau + d\tau$. Then P_{-++-} becomes the joint probability of an upward crossing in $(t, t + dt)$ and a downward crossing in $(t + \tau, t + \tau + d\tau)$. If we divide by $\frac{1}{2}\beta dt$, the probability of an upward crossing in $(t, t + dt)$, the conditional probability $Q(\tau)d\tau$ is obtained. Thus

[21] For a detailed derivation in n variables, see J. A. McFadden, "Urn models of correlation and a comparison with the multivariate normal integral," *Ann. Math. Statist.*, vol. 26, pp. 478–489; September, 1955. See sec. 6.

[22] Ibid, bibliography.

$$Q(\tau)\, d\tau = \frac{P_{-++-}}{\frac{1}{2}\beta\, dt}. \tag{85}$$

In order to utilize this result, P_{-++-} must be expanded in a double Taylor series in dt and $d\tau$. All terms are retained up to the second order. By differentiating under the integral sign in the time average,

$$w(\tau_1, \tau_2, \tau_3) = \lim_{L\to\infty} \frac{1}{2L} \cdot \int_{-L}^{L} x(t)x(t+\tau_1)x(t+\tau_2)x(t+\tau_3)\, dt, \tag{86}$$

it is found that

$$\left(\frac{\partial w}{\partial \tau_1}\right)_{0+,\tau,\tau+} = \left(\frac{\partial w}{\partial \tau_3}\right)_{0+,\tau,\tau+} = r'(0+), \tag{87}$$

$$\left(\frac{\partial^2 w}{\partial \tau_1^2}\right)_{0+,\tau,\tau+} = \left(\frac{\partial^2 w}{\partial \tau_3^2}\right)_{0+,\tau,\tau+} = r''(0+), \tag{88}$$

and furthermore,

$$w(0+, \tau, \tau+) = 1. \tag{89}$$

The result of the detailed expansion is that all terms of P_{-++-} cancel out except the one proportional to $dtd\tau$; then (85) yields the result

$$Q(\tau) = \frac{1}{8\beta}\left[\left(\frac{\partial^2 w}{\partial \tau_1\, \partial \tau_3}\right)_{0+,\tau,\tau+} + r''(\tau)\right]. \tag{90}$$

By comparison of (8) and (90), the final result, which is (11), is reached.

By differentiation of (86), it can be shown that $U(\tau)$ in proportional to the autocorrelation function of the product $x(t)x'(t+)$.

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Recursion Formulas for Growing Memory Digital Filters*

MARVIN BLUM†

Summary—**A growing memory digital filter is defined by considering the input (y_{v-u})—output (Z_m) relationship in the form $Z_m = \sum_{u=0}^{m} W_{um}\, y_{m-u}$, $m = 0, 1, 2, \cdots$ where W_{um} is the weighting sequence of a linear time varying digital filter. Contained herein are a derivation of an optimum growing memory smoothing and prediction filter in the least squares sense for polynomial input functions, (of degree $= K$) and a theorem on the class of time invariant sequence W_u, which are solutions of a difference equation of finite order, and an application of the theorem to the synthesis of sampled correlated noise by digital processes, using recursion formulas. The recursion formulation represents a practical solution to the generation of a correlated noise sequence on line during simulation studies on digital computers.**

Introduction

In a previous paper by the author,[1] the solution of the fixed memory least squares filter, using recursion methods, was presented. This part of the paper is an extension of the derivation and so the notation will be written with a minimum of explanation.

For the fixed memory filter operating over an interval $(n-1)T$, *i.e.*, the n most recent data points, the output of the filter at time mT can be written in the form

$$Z_m = \sum_{u=1}^{n} C_u y_{m+n-u}. \tag{1}$$

The action of the filter is to take a sliding weight of the n most recent data points as m is increased by one.

The output Z_m will be an errorless estimate for input polynomials up to and including degree K in the absence of noise. The desired output is taken as the Lth derivative of the input evaluated at $(m+\alpha)T$. The equation for the coefficients C_u is given in Appendix I by (47). The functions $\xi_{v,n}(u)$ are the orthogonal polynomials 2, 3 and $S(v, n)$ is the sum of squares given by (9). The quantity $\xi_{v,n}^{L}(n+\alpha)$ is obtained from the more general definition:

$$\left.\frac{d^a}{du^a}\xi_{v,n}(u)\right|_{u=b} \equiv \xi_{v,n}^{(a)}(b). \tag{2}$$

Instead of considering a filter with a fixed memory, one can let n change by one for each new sample. Thus when three samples are available one may fit, say, a least squares straight line to the 3-data points and obtain estimates from the 3-point curve fit. When another data point is sampled, one may now fit a least squares

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[1] M. Blum, "Fixed memory least squares filters using recursion methods," IRE Trans. on Information Theory, vol. 3, pp. 178–182; September, 1957.

straight line to the 4-data points and obtain estimates from the 4-point curve fit. This procedure may be repeated with the next sample to obtain a 5-point curve fit and so on. The property of such a filter is that the memory span grows with each new data point and the estimates are more accurate because they are based on more data. This increase in accuracy is obtained provided the coefficients of the polynomial remain invariant over the interval being sampled. In the fixed memory filter one requires only that the function be representable by a polynomial over the interval of prediction and memory span of the filter. The coefficients of the polynomial may change from interval to interval provided the degree of the polynomial does not increase.

The output of the growing arc filter can be written from a modification of (1) as:

$$Z_n = \sum_{u=1}^{n} C_u y_u \qquad n = K+1, K+2, \cdots \tag{3}$$

where the coefficients C_u are given by (47) of Appendix I. For the sake of simplicity the sampling interval T is taken equal to one. To obtain the value of Z_n for $T \neq 1$, multiply C_u by $| T^{-L} |$.

Recursion Formulas for Growing Memory Filters

One procedure for obtaining a growing memory filter is to compute for each value of n the array of coefficients C_u and utilize (3). This may be practical if one is interested in relatively small values of n. One may precompute and store the values and use them as the data are obtained. As time progresses and n becomes larger, this procedure becomes burdensome and so an alternate method using recursion procedures will be considered. For purposes of convenience the time origin has been labeled 1 and the data are taken from 1 to n. The time origin can be arbitrarily set to correspond to the first acquisition of data without restricting the generality of the solution.

Let the Wth moment be defined as

$$\sum_{u=1}^{n} y_u(u)^W \equiv M(W, n) \tag{4}$$

then the Wth moment satisfies the recursion relationship

$$M(W, n) = M(W, (n-1)) + y_n(n)^W \qquad W = 0, 1, 2, \cdots K. \tag{5}$$

Define

$$\phi_{K,n,W}^{(L)} = \sum_{v=W}^{K} \frac{(-1)^{W-v}}{W!} \frac{\xi_{v,n}^{(W)}(n+1)\xi_{v,n}^{(L)}(n+\alpha)}{S(v,n)} \tag{6}$$

or

$$\phi_{K,n,W}^{(L)} = \sum_{v=W}^{K} \frac{1}{W!} \frac{\xi_{v,n}^{(W)}(0)\xi_{v,n}^{(L)}(n+\alpha)}{S(v,n)}. \tag{7}$$

Then (see Appendix I)

$$Z_n = \sum_{W=0}^{K} M(W, n)\phi_{K,n,W}^{(L)}. \tag{8}$$

Recursion Formulas for $\xi_{v,n}^{W}(u)$ and $S(v,n)$

In a paper by Allen[2] it is shown that

$$S(v, n) = \frac{\{(v!)^4\}\left\{n \prod_{j=1}^{v} (n^2 - j^2)\right\}}{(2v)!(2v+1)!} \equiv \sum_{u=1}^{n} [\xi_{v,n}(u)]^2 \tag{9}$$

and therefore satisfies the recursion formula.

$$S(v, n) = \left\{\frac{n+v}{n-(v+1)}\right\} S(v,(n-1)). \tag{10}$$

When $v + 1/n \ll 1$ then

$$S(v, n) \cong S[v, (n-1)]\left\{1 + \frac{2v+1}{n}\right\}. \tag{11}$$

A recursion formula for the orthogonal polynomials is given by Anderson and Hauseman[3] as:

$$\xi_{v+1,n}(u) \equiv \xi_{1,n}(u)\xi_{v,n}(u) - \frac{v^2(n^2-v^2)}{4(4v^2-1)} \xi_{v-1,n}(u), \qquad v = 0, 1, 2 \cdots \tag{12}$$

where

$$\xi_{0,n} = 1 \tag{13}$$

and

$$\xi_{1,n} = (u - \bar{u}), \qquad \bar{u} = \frac{n+1}{2}. \tag{14}$$

The recursion formula is an identity in u considered as a continuous variable $-\infty \leqq u \leqq +\infty$ so that (12) may be differentiated on both sides giving

$$\frac{d^Z}{du^Z} \xi_{v+1,n}(u) = \xi_{1,n}(u) \frac{d^Z}{du^Z} \xi_{v,n}(u) + Z \frac{d^{Z-1}}{du^{Z-1}} \xi_{v,n}(u) - \frac{v^2(n^2-v^2)}{4(4v^2-1)} \frac{d^Z}{du^Z} \xi_{v-1,n}(u), \qquad Z = 0, 1, 2 \cdots, \quad v = 0, 1, 2 \cdots. \tag{15}$$

In particular when $Z = W$, $u = 0$

$$\xi_{v+1,n}^{(W)}(0) = -\frac{(n+1)}{2} \xi_{v,n}^{(W)}(0) + W\xi_{v,n}^{(W-1)}(0) - \frac{v^2(n^2-v^2)}{4(4v^2-1)} \xi_{v-1,n}^{(W)}(0) \tag{16}$$

and when $Z = L$, and $u = n + \alpha$ then

$$\xi_{v+1,n}^{(L)}(n+\alpha) = \frac{(n+2\alpha-1)}{2} \xi_{V,n}^{(L)}(n+\alpha) + L\xi_{v,n}^{(L-1)}(n+\alpha) - \frac{v^2(n^2-v^2)}{4(4v^2-1)} \xi_{v-1,n}^{(L)}(n+\alpha). \tag{17}$$

[2] F. E. Allen, "The general form of the orthogonal polynomials for simple series, with proofs of their simple properties," *Proc. Roy. Soc., Edinburgh*, vol. 50, pp. 310–320; 1935.

[3] R. L. Anderson and E. E. Hauseman, "Tables of Orthogonal Polynomial Values Extended to $N = 104$," Agricultural Experiment Station, Iowa State College of Agricultural and Mechanical Arts, Ames, Iowa, Res. Bull. 297; April, 1942.

Eq. (8) can be generalized in matrix form if one lets the parameter $L = 0, 1, 2, 3 \cdots K$. Then one may obtain an output consisting of the predicted value of the input variable at $(n + \alpha)T$ and all of its derivatives up to the Kth derivative at $(n + \alpha)T$.

In computing the estimate of the Kth derivative using the various recursion relationships, the terms required to obtain estimates of all the derivatives up to K are available as intermediate results.

A Theorem on Growing Memory Digital Filters

Let the input to a linear time invariant digital filter be the sequence x_u and the output, the sequence y_m, then the relationship between the input and the output is given by:[4]

$$y_m = \sum_{u=0}^{\infty} W_u x_{m-u} \tag{18}$$

where W_u is the weighting sequence of the filter. Define a class of sequences W_u such that W_u is a solution to a linear difference equation with constant (d_k) coefficients of order n' and let this class be noted by $H_{n'}$, *e.g.*,[5]

$$\sum_{k=0}^{n'} \{d_k D^k\} W_u = 0, \qquad u = 0, 1, \cdots \tag{19}$$

where D^k is the advance operator defined by

$$D^k\{W_u\} \equiv W_{u+k} \qquad \begin{matrix} u = 0, 1, 2, \cdots \\ k = 0, 1, 2, \cdots \end{matrix} . \tag{20}$$

Then the following theorem for the class $W_u \, \varepsilon \, H_{n'}$ will be shown.

Theorem: If $x_u = 0$ for $u < 0$, then for $W_u \, \varepsilon \, H_{n'}$

$$y^*_{n'+v} = \sum_{u=0}^{n'+v} W_u x_{n'+v-u},$$

$$v = -n', -n' + 1, \cdots 0, 1, \cdots \tag{21}$$

$$y^*_{n'+v} = \sum_{k=0}^{n'-1} - d_k y^*_{v+k} + g_k x_{v+n'-k} \tag{22}$$

where

$$g_k = \sum_{u=0}^{k} d_{n'-u} W_{k-u}. \tag{23}$$

Corollary: If $x_u \neq 0$ for $u < 0$ then the output obtained from the digital filter $W_u \, \varepsilon \, H_{n'}$ on the assumption $x_u = 0$, $u < 0$, as given by (21), is in error by an amount given by

$$\xi_{n'+v} = y_{n'+v} - y^*_{n'+v} = \sum_{u=n'+v+1}^{\infty} W_u x_{n'+v-u}. \tag{24}$$

The proof of the theorem is given by Appendix II.

[4] H. M. James, N. B. Nichols, and R. S. Phillips, "Theory of Servomechanisms," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y. vol. 25; 1947.

[5] C. Jordan, "Calculus of Finite Differences," Chelsea Publishing Co., New York, N. Y.; 1950.

Derivation of the Weighting Sequence $W_u \varepsilon H_{n'}$

Consider a continuous linear time invariant filter whose transfer function is given by[4]

$$Y(p) = \frac{H(p)}{G(p)} \tag{25}$$

where $H(p)$ is a polynomial of degree h, and $g(p)$ is a polynomial of degree n' with $n' > h$. Then the corresponding impulse response is given by the inverse Laplace transform $W(t)$ and is of the form

$$W(t) = \sum_{\kappa=1}^{a} \sum_{j_\kappa=1}^{S_\kappa} \frac{C_{j_\kappa} t^{j_\kappa - 1} e^{p_\kappa t}}{(j_\kappa - 1)!}, \qquad t \geqq 0 \tag{26}$$

where the $p_1, p_2, \cdots p_a$ are roots of $G(p) = 0$ and the root p_K is of multiplicity S_K.

A corresponding digital filter is obtained from defining the weighting sequence as

$$W_u = TW(uT), \qquad u = 0, 1, \cdots . \tag{27}$$

Substituting (27) into (26) one may write the following difference equation on W_u:

$$\prod_{j=1}^{a} \{(D - Z_j)^{S_j}\} W_u = 0 \tag{28}$$

where

$$Z_j = e^{Tp_j}, \qquad j = 1, 2, \cdots a. \tag{29}$$

If $Y(p)$ is a stable filter then real part of $p_j \equiv R\{p_j\} < 0$, $j = 1, 2, \cdots a$.

Expanding (28) one sees the $W_u \, \varepsilon \, H_{n'}$. It is also evident that the class of weighting sequences $W_u \, \varepsilon \, H_{n'}$, is derived from the continuous filters which are representable as the ratio of rational polynomials in p when expressed in terms of their transfer function.

Noise Synthesis

The problem considered is as follows: let the output of the continuous filter whose input is white noise be sampled every T units. Simulate the entire process by digital means.

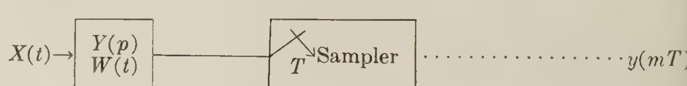

Fig. 1—Continuous white noise passed through stable linear filter $Y(p)$ and sampled every T seconds.

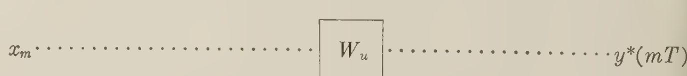

Fig. 2—Sampled white noise passed through stable linear digital filter whose output has the same statistical characteristics as $y(mT)$.

The analog can be seen from Figs. 1 and 2 in which the combined effects of passing white noise through a filter and sampling every T units is simulated digitally by operating on the uncorrelated sequence (spaced T units apart) with the digital filter whose weighting sequence is W_u.

The synthesis procedure will be summarized and recursion equations of the form (22) will be derived for two cases.

Synthesis Procedure

1) Fix the exact form of the continuous filter $Y(p)$.
2) Determine the roots p_j of $G(p) = 0$.
3) Solve for the impulse response $W(t)$ using the inverse Laplace transform of $Y(p)$.
4) Determine the form of the difference equation in which $W_u = TW(uT)$ satisfies from (28).
5) Expand (28), to obtain the coefficients $d_{n'-u}$.
6) Compute W_{k-u} from 3) and 4).
7) Substitute 5) and 6) into (23) and then into (22).

Example I

Simple Pair of Complex Roots

Let

$$Y(p) = \frac{C}{(p - p_1)(p - p_1^*)} \tag{30}$$

where C has the dimensions t^{-2}, therefore $a = 2 = n$, $S_1 = 1$, and $S_2 = 1$. Let the real p_1 = real $p_1^* < 0$. The impulse response of $Y(p)$ is given by the inverse Laplace transform

$$W(t) = \frac{1}{2\pi j}\int_{b'-j\infty}^{b'+j\infty} dp\, Y(p) e^{+pt} \tag{31}$$

so that

$$W(t) = C\,\frac{e^{p_1 t} - e^{p_1^* t}}{p_1 - p_1^*} = \frac{C}{B}\sin Bte^{\alpha t}, \tag{32}$$

where

$$\left.\begin{aligned} p_1 &\equiv \alpha + jB,\\ p_1^* &= \alpha - jB.\end{aligned}\right\} \tag{33}$$

and

Then by (27)

$$W_u = \frac{CT}{B}e^{\alpha T u}\sin uBT, \qquad u = 0, 1, 2, \cdots, \tag{34}$$

and

$$Z_1 = e^{(\alpha + jB)T} = e^{\alpha T}(\cos BT + j\sin BT), \tag{35}$$

$$Z_2 = e^{(\alpha - jB)T} = e^{\alpha T}(\cos BT - j\sin BT).$$

To determine the d_k one forms

$$\begin{aligned}(D - Z_1)(D - Z_2) &= D^2 - (Z_1 + Z_2)D + Z_1 Z_2,\\ &= d_2 D^2 + d_1 D + d_0\end{aligned} \tag{36}$$

so that from (35) and (36) one obtains

$$\left.\begin{aligned} d_2 &= d_n = 1,\\ d_1 &= -2e^{\alpha T}\cos BT,\\ d_0 &= e^{2\alpha T}\end{aligned}\right\}. \tag{37}$$

The unknown coefficients g_0 and g_1 are obtained from (23) as:

$$\left.\begin{aligned} g_0 &= W_0 = 0,\\ g_1 &= W_1 d_2 + W_0 d_1 = W_1 = \frac{CT}{B}(\sin BT)e^{\alpha T}\end{aligned}\right\}. \tag{38}$$

Substituting into (22) one obtains the desired recursion formula.

$$\begin{aligned} y_{2+v}^* &= (2q\cos BT)y_{v+1}^* - q^2 y_v^* + \frac{CT}{B}q\sin BT\,\{x_{v+1}\}\\ v &= -2, -1, 0, 1, 2, \cdots,\end{aligned} \tag{39}$$

where, $q \equiv e^{\alpha T}$ and $y_m^* = x_m = 0$ for $m < 0$.

One may verify by substitution that the value of y_{2+v}^* generated by the recursion formula is identical with

$$y_{2+v}^* = \sum_{u=0}^{2+v} W_u x_{2+v-u} \tag{40}$$

on the assumption that $x_m = 0$ for $m < 0$, $v = -2, -1, 0, 1, 2, 3, \cdots$.

Second-Order Low-Pass Filter

Let

$$Y(p) = \frac{c}{(p - p_1)^2}, \qquad p_1 = \alpha \quad \text{and}$$

$$Z_1 = e^{\alpha T} = q, \tag{41}$$

then

$$(D - Z_1)^2 = D^2 - 2DZ_1 + Z_1^2.$$

From (41) one obtains

$$\left.\begin{aligned} d_2 &= d_n = 1,\\ d_1 &= -2q,\\ d_{2\to 0} &= q^2\end{aligned}\right\}. \tag{42}$$

The impulsive response of the filter is given by

$$W(t) = \frac{1}{2\pi j}C\int_{b'+j\infty}^{b'-j\infty}\frac{1}{(p - p_1)^2}e^{+pt}\,dt,$$

$$W(t) = Cte^{p_1 t}.$$

Therefore, by (27) the weighting sequence is given by

$$W_u = CT^2 uq^u.$$

Using (23) one obtains

$$\left.\begin{aligned} g_0 &= d_2 W_0 = 0,\\ g_1 &= d_2 W_1 + d_1 W_0 = W_1 = (CT^2)q\end{aligned}\right\}. \tag{43}$$

From (22) one obtains

$$y_{v+2}^* = -(d_0 y_v^* + d_1 y_{v+1}^*) + g_0 x_{v+2} + g_1 x_{v+1}. \tag{44}$$

Substituting (42) and (43) into (44) one obtains the desired recursion formula

$$y_{v+2}^* = +2qy_{v+1}^* - q^2 y_v^* + (CT^2)qx_{v+1}. \tag{45}$$

Error Analysis

At this point it is required to distinguish between two conditions on the input noise. For condition A the noise is switched on at $t = 0$, and one may assume that $x_u = 0$, $u < 0$, or is interested in the solutions under this condition. For this case the output of the digital filter using the recursion theorem is an identity. One may consider unstable filters of the form (25). The mean square of the output noise will tend to infinity as $u \to \infty$ if $Y(p)$ is unstable.

For condition B the noise is considered to exist from the present time to minus infinity. The use of the recursion formula assumes the $x_u = 0$, $u < 0$. This assumption causes a transient error as given by (24). If $Y(p)$ is a stable filter, then the transient error ε_m approaches zero as $m \to \infty$ as $0 \mid e^{\bar{p}Tm} \mid$ where p is the root of $G(p)$ whose magnitude is closest to zero. Similarly all the statistical properties of y_m^* approach y_m as $m \to \infty$. Such properties as the autocorrelation function and power spectrum of the sequence y_m^* approach the same functions of the sequence y_m as $0 \mid e^{2\bar{p}Tm} \mid$ as $m \to \infty$. Thus to simulate the sampled output of a filter $Y(p)$ for white noise input under condition B, it is required that $Y(p)$ be stable and that one use the steady state output of the digital filter. Thus the process is started at $m = 0$; however, the representative noise sequence is not used until $m \geqq M$, where M is selected to satisfy some accuracy criteria.

Generating Sampled White Noise

A white noise source is characterized by having a constant power spectrum over all frequencies (f). If the spectrum is so defined that

$$\int_0^\infty A(f)\, df = \sigma^2 \equiv E(\chi^2)$$

where σ^2 is the mean square of the noise. Then for white noise, σ^2 is infinite. For sampled noise the power spectrum is periodic in f with period $1/T$. The mean square error for the sampled noise is given by

$$\sigma_T^2 = \int_0^{1/2T} df A(f)$$

where[4]

$$A(f) = 4T\left[\frac{R(0)}{2} + \sum_{b=1}^{\infty} R(bT) \cos 2\pi f T b\right]$$

and

$$R(bT) \equiv E(\chi_t \chi_{t+|b|}).$$

For uncorrelated sampled noise

$$R(bT) = \sigma_T^2, \qquad b = 0$$
$$= 0 \qquad b \neq 0$$
$$\therefore A(f) \equiv G_0 = 2T\sigma_T^2.$$

Therefore, given a white noise source whose power spectrum constant amplitude vs frequency is G_0, the simulation of the sampled white noise with sampling interval equal to T is obtained then with an uncorrelated random sequence x_u with mean square given by $G_0/2T$. A method of generating a pseudo random sequence which is uncorrelated, distributed normally and capable of being scaled to the desired mean square is given by Kahn.[6] This sequence is used as the input sequence for (22). The output sequence $y_{n'+v}^*$ for $n' + v \geqq M$ then has the desired statistical properties.

The method is summarized in Appendix III.

Appendix I

The input-output relationship of the digital filter is given by

$$Z_n = \sum_{u=1}^{n} C_u(K, n, \alpha) y_u. \tag{46}$$

Since[7]

$$C_u(K, n, \alpha) = \sum_{v=L}^{K} \frac{\xi_{v,n}(u)\xi_{v,n}^{(L)}(n + \alpha)}{S(v, n)}, \tag{47}$$

then

$$Z_n = \sum_{u=1}^{n} \sum_{v=L}^{K} \frac{y_u \xi_{v,n}(u)\xi_{v,n}^{(L)}(n + \alpha)}{S(v, n)}. \tag{48}$$

One may expand $\xi_{v,n}(u)$ about $(u - \bar{u})$ and obtain the identity

$$\xi_{v,n}(u) \equiv \sum_{a=0}^{v} H_{a,v}(u - \bar{u})^a, \tag{49}$$

where

$$\bar{u} = \frac{n + 1}{2}. \tag{50}$$

Let the quantity $(u - \bar{u})^a$ be expanded in powers of u. Then

$$(u - \bar{u})^a = \sum_{b=0}^{a} u^{a-b}(-\bar{u})^b \binom{a}{b}, \tag{51}$$

so that substituting (51) and (49) into (48) one obtains,

$$Z_n = \sum_{u=1}^{n} \sum_{v=L}^{K} \sum_{a=0}^{v} \sum_{b=0}^{a} \frac{y_u (u)^{a-b}(-\bar{u})^b \binom{a}{b} H_{a,v}\xi_{v,n}^{(L)}(n + \alpha)}{S(v, n)}. \tag{52}$$

Let $(a - b) = W$, then collecting like powers of u, e.g., the coefficient of u^W, one obtains:

$$Z_n = \left\{\sum_{u=1}^{n} y_u u^W\right\} \cdot \left\{\sum_{v=W}^{K} \sum_{a=W}^{v} \frac{(-\bar{u})^{a-W}\binom{a}{a - W} H_{a,v}\xi_{v,n}^{(L)}(n + \alpha)}{S(v, n)}\right\}. \tag{53}$$

[6] H. Kahn, "Application of Monte Carlo," Rand Corp., Santa Monica, Calif., Res. Memo. RM-1237-AEC; April 19, 1954. Revised, April 27, 1956.

[7] M. Blum, "An extension of the minimum mean square prediction theory for sampled input data," IRE Trans. on Information Theory, vol. 2, pp. 176–184; September, 1956.

Now

$$\sum_{=W}^{v} (-\bar{u})^{a-W}\binom{a}{a-W}H_{a,v} = \frac{(-1)^{v+W}}{W!}\frac{d^W}{du^W}\xi_{v,n}(u)\bigg|_{n+1}$$

$$\equiv \frac{(-1)^{v+W}}{W!}\xi_{v,n}^{(W)}(n+1) \qquad (54)$$

or

$$\sum_{=W}^{v} (-\bar{u})^{(a-W)}\binom{a}{a-W}H_{a,v} = \frac{1}{W!}\frac{d^W}{du^W}\xi_{v,n}(u)\bigg|_{u=0}$$

$$\equiv \frac{1}{W!}\xi_{v,n}^{(W)}(0).$$

Then defining

$$M(W, n) = \sum_{u=1}^{n} y_u \cdot u^W \qquad (55)$$

and

$$\phi_{K,n,W}^{(L)} = \sum_{v=W}^{K} \frac{1}{W!}\frac{\xi_{v,n}^{(W)}(0)\xi_v^{(L)}(n+\alpha)}{S(v,n)} \qquad (56)$$

one obtains

$$Z_n = \sum_{W=0}^{K} M(W,n)\phi_{K,n,W}^{(L)}.$$

Appendix II

Proof of Theorem

It will be shown that the difference equation which defines the recursion procedure is the equivalent of computing

$$y_m^* = \sum_{u=0}^{m} W_u x_{m-u} \qquad (57)$$

provided the weighting sequence W_u satisfies the same complimentary difference equation as the y_m^* do. This condition holds for the class of functions to which the method is restricted and for the weighting sequence as defined in (26) and (27).

Proof

Dropping the prime on n', let

$$m = n + v.$$

Then (22) may be written

$$y_m^* = -\sum_{k=0}^{n-1} d_k y_{m-n+k}^* + \sum_{j=0}^{n-1} g_j x_{m-j}. \qquad (58)$$

From (23) one has

$$g_j = \sum_{v=0}^{j} d_{n-v} W_{j-v}, \qquad (59)$$

and from (18) one has, for $x_m = 0$, $m < 0$,

$$y_m^* = \sum_{u=0}^{m} W_u x_{m-u}. \qquad (60)$$

Substituting (60) and (59) into (58) one obtains

$$y_m^* = -\sum_{k=0}^{n-1}\sum_{u=0}^{m-n+k} d_k W_u x_{m-n+k-u} + \sum_{j=0}^{n-1}\sum_{v=0}^{j} d_{n-v}W_{j-v}x_{m-j}. \qquad (61)$$

Let $m \geqq n$ and $m-(n-1) = L$, $L \geqq 0$ and

$$m - n + k - u = L - b, \qquad b = 1, 2, \cdots L.$$

Since the smallest index of x of the second summation is $x_{m-(n-1)} = x_L$, then the coefficient of x_{L-b} is given by the first double summation only and is

$$-\sum_{k=0}^{n-1} d_k W_{(m-n+k)+(b-L)} = W_{m-(L-b)}, \qquad (62)$$

since the W_m satisfy the same complimentary difference equation as the y_m^*. The coefficients of the X_u when $u \geqq L$ are given by the sum of the two terms of (61) as follows: Let $(m-n+k-u) = L+c$, $c = 0, 1, 2, 3, \cdots, m-L$, then the coefficient of x_{L+c} from the first term of (61) is given by

$$(-)\sum_{k=0}^{n-1} d_k W_{m-n+k-(L+c)}, \qquad (63)$$

and the coefficient of $x_{(L+c)}$ from the second term of (61) is given by

$$\sum_{v=0}^{m-(L+c)} d_{n-v}W_{m-(L+c+v)}. \qquad (64)$$

Combining (63) and (64) one obtains

$$x_{L+c}\left\{\sum_{v=0}^{m-(L+c)} d_{n-v}W_{m-(L+c+v)} - \sum_{k=0}^{n-1} d_k W_{m+k-(L+c+n)}\right\} = x_{L+c}W_{m-(L+c)}. \qquad (65)$$

Since all terms cancel by subtraction except the coefficient of d_n in the first summation, one will note that $d_n = 1$. The combined results of (65) and (62) give the desired result

$$y_m^* = \sum_{u=0}^{m} W_u X_{m-u}.$$

Appendix III

Generation of "Sampled" White Noise[6]

Let $\bar{R}_i = (5^5)^i$, $\bar{R}_{i+j} = \bar{R}_i \times \bar{R}_j$. r_i = remainder after dividing R_i by 2^{35}, and let r_i be scaled from 0 to 1.

The approximate statistical properties of r_i are as follows:

1) r_i is a random variable uniformly distributed in the interval 0 to 1.
2) $E(r_i) = 1/2$ (average value of r_i).
3) Variance $r_i = E(r_i - E(r_i))^2 = 1/12$.
4) The r_i's are completely random, *e.g.*, $E(r_i r_{i+j}) = 1/4\ \delta_{0j}$ where δ_{jk} is the Kronecker Delta, $\delta_{jk} = 1$, $j = k$, $\delta_{jk} = 0$, $j \neq k$.

For most physical problems, the probability distribution of the noise is Gaussian. The probability dis-

tribution of r_i is not suitable for representing the noise encountered in physical systems. However, by a simple weighted addition of the r_i's one may generate a variable N_i which is almost Gaussian. In fact, by adding a sufficient number of r_i's one may form a variable N_i whose distribution is arbitrarily close to a normal distribution. This follows from the central limit theorem of statistics.

Let $x_1, x_2 \cdots x_L$ be L independent samples from a distribution $f(x)dx$ and let the mean value of x be zero, the variance be unity, and

$$Z = \frac{1}{(L)^{1/2}} \sum_{i=1}^{L} x_i.$$

In the limit as $L \to \infty$ the random variable Z is normally distributed with zero mean and unity variance. Consider the variable[8]

$$N_j = \sqrt{\frac{12}{L}} \sum_{k=0}^{L-1} (r_{Lj+k} - 1/2) \qquad j = 0, \pm 1, \pm 2, \pm 3 \cdots .$$

[8] If the right-hand side is multiplied by λ, then the variance will be λ^2.

Note that the N_j's are formed from nonoverlapping sums of the r_j's. Then in the limit, as $L \to \infty$, N_j will be normally distributed with zero mean and unity variance. For all values of L, N_j will have a zero mean and unity standard deviation. For L sufficiently large N_j will be approximately Gaussian distributed.

Properties of N_j

1) $E(N_j) = 0$.
2) Var $(N_j) = 1$.
3) $E(N_j N_{j+k}) = \delta_{0k}$ (completely random).
4) For large L, N_j is approximately Gaussian distributed.
5) N_j is truncated since $|N| \leqq \sqrt{3L}$ and since the standard deviation of N is 1. Then, if $L = 6$, N can vary between ± 4.2 standard deviations. It is felt that 6 is probably a lower limit to L since, if N is truncated too severely, the effects of occasional extreme deviations will be masked by the truncation effect.

The Fluctuation Rate of the Chi Process*

RICHARD A. SILVERMAN†

Summary—The chi process is defined as a natural generalization of the chi distribution of statistical theory. A formula is derived for the expected number of level crossings per second of the chi process. The formula contains as a special case the familiar expression for the fluctuation rate of the envelope of Gaussian noise.

Introduction

The expected number of times per second that a stationary random process $\xi(t)$ assumes the value α is given by the familiar expression[1]

$$N_\xi(\alpha) = \int_{-\infty}^{\infty} |y| \, p(\alpha, y) \, dy, \tag{1}$$

where $p(x, y)$ is the bivariate probability density of $\xi(t)$ and its time derivative $\dot{\xi}(t)$, taken at any fixed time t. Consider first the case where $\xi(t)$ denotes a zero-mean Gaussian process $g(t)$, with correlation function

$$R(\tau) = \frac{\langle g(t)g(t + \tau)\rangle}{\langle g^2(t)\rangle} = \int_0^\infty \Phi(\omega) \cos \omega\tau \, d\omega.$$

Here the angular parentheses denote the expectation or ensemble average, and $\Phi(\omega)$ is the (normalized) power spectrum of the process $g(t)$. Then (1) reduces to[2]

$$N_g(\alpha) = \sqrt{\frac{2}{\pi}} \sigma \sqrt{-R_0''} W_g(\alpha), \tag{2}$$

where $\sigma^2 = \langle g^2(t)\rangle$, R_0'' is the second derivative[3] of $R(\tau)$ with respect to τ, evaluated at $\tau = 0$, and

$$W_g(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-x^2/2\sigma^2)$$

is the univariate probability density[4] of the process $g(t)$, *i.e.*, the probability density of a Gaussian random variable with mean zero and variance σ^2. Alternatively,

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[1] S. O. Rice, "Mathematical analysis of random noise," reprinted in the collection "Selected Papers on Noise and Stochastic Processes" (N. Wax, ed.), Dover Publications, Inc., New York, N. Y., p. 190; 1954.

[2] *Ibid.*, p. 193.

[3] The prime will be used to denote differentiation with respect to τ, and the dot to denote differentiation with respect to t. We assume that $g(t)$ and the components $g_i(t)$ of the chi process (see following) have correlation functions which are twice differentiable at the origin.

[4] By the univariate probability density of a stationary random process $\xi(t)$, we mean the probability density of the random variable $\xi(t_0)$, where t_0 is any fixed value of t.

(2) can be written as

$$N_g(\alpha) = \sqrt{\frac{2}{\pi}}\,\sigma\sqrt{\overline{\omega^2}}\,W_g(\alpha), \qquad (3)$$

where

$$-R_0'' = \overline{\omega^2} = \int_0^\infty \omega^2\Phi(\omega)\,d\omega > 0$$

is the mean square (angular) frequency of the process $g(t)$.

Consider next the case where $\xi(t)$ denotes the envelope $A(t)$ of the Gaussian process $g(t)$. Then, as shown by several authors,[5–7] (1) reduces to

$$N_A(\alpha) = \sqrt{\frac{2}{\pi}}\,\sigma\,\Delta\omega\,W_2(\alpha), \qquad (4)$$

where

$$\begin{aligned} W_2(x) &= (x/\sigma^2)\exp(-x^2/2\sigma^2), & x \geq 0,\\ W_2(x) &= 0 \qquad\qquad\qquad\quad , & x < 0, \end{aligned} \qquad (5)$$

is the univariate probability density of $A(t)$, *i.e.*, the familiar Rayleigh distribution. The quantity $\Delta\omega$ is given by

$$(\Delta\omega)^2 = \overline{\omega^2} - (\bar{\omega})^2, \qquad (6)$$

where $\overline{\omega^2}$ is the mean square frequency of the underlying Gaussian process $g(t)$, and

$$\bar{\omega} = \int_0^\infty \omega\Phi(\omega)\,d\omega$$

is its mean frequency. The similarity of (3) and (4) is apparent. It follows at once from (3), (4), and (6) that any two of the quantities $\bar{\omega}$, $N_g(\alpha)$ and $N_A(\alpha)$ determine the other.

In this note we define an infinite class of random processes, the so-called chi processes. We find that the expression for the fluctuation rate of these processes is a natural generalization of (2), (3), and (4).

The Chi Process and its Fluctuation Rate

We define the chi process by generalizing to random processes the familiar definition of the chi distribution.[8]

Definition

Let $g_1(t), \cdots, g_n(t)$ be n independent stationary Gaussian random processes with zero means and such that

$$\langle g_i(t)g_i(t+\tau)\rangle = \sigma^2R(\tau), \qquad 1 \leq i \leq n. \qquad (7)$$

Then by the chi process with n degrees of freedom is meant the random process

$$\chi_n(t) = \left[\sum_{i=1}^n g_i^2(t)\right]^{1/2}. \qquad (8)$$

The processes $g_i(t)$, $1 \leq i \leq n$ can be regarded as the components of a vector random process which has the process $\chi_n(t)$ as its length. This fact can be summarized by calling the processes $g_i(t)$, $1 \leq i \leq n$, the *components* of $\chi_n(t)$. The conditions (7) mean that the components of $\chi_n(t)$ are identically autocorrelated. The univariate distribution of the process $\chi_n(t)$ is, of course, the chi distribution with n degrees of freedom, for which the probability density is[8]

$$\begin{aligned} W_n(x) &= \frac{2}{2^{n/2}\sigma^n\Gamma(n/2)}x^{n-1}\exp(-x^2/2\sigma^2), & x \geq 0,\\ W_n(x) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad , & x < 0. \end{aligned} \qquad (9)$$

[Note the consistency of (5) and (9).] Processes of this type have also been studied by Miller, *et al.*, who call them *generalized Rayleigh processes.*[9,10]

We now derive an expression for the fluctuation rate of the chi process.

Theorem

The expected number of times per second that the process $\chi_n(t)$ assumes the value α is given by

$$N_n(\alpha) = \sqrt{\frac{2}{\pi}}\,\sigma\sqrt{-R_0''}\,W_n(\alpha). \qquad (10)$$

Proof

We consider first the trivial but exceptional case $n = 1$, and note that $\chi_1(t) = \sqrt{g^2(t)} = |g(t)|$. Since $|g(t)| = \alpha$ if and only if $g(t) = \alpha$ or $g(t) = -\alpha$, it follows from (2) that

$$\begin{aligned} N_1(\alpha) &= \sqrt{\frac{2}{\pi}}\,\sigma\sqrt{-R_0''}\,2W_g(\alpha), & \alpha \geq 0,\\ N_1(\alpha) &= 0 \qquad\qquad\qquad\qquad , & \alpha < 0. \end{aligned}$$

Since by (9)

$$\begin{aligned} W_1(\alpha) &= 2W_g(\alpha), & \alpha \geq 0,\\ W_1(\alpha) &= 0 \qquad\quad , & \alpha < 0, \end{aligned}$$

(10) is established for the case $n = 1$. Note that unlike the case $n \geq 2$, $\chi_1(t)$ assumes the value zero (without passing through it) at a nonzero rate.

When $n \geq 2$, we apply the basic formula (1), which requires that we first calculate the bivariate probability density of $\chi_n(t)$ and its derivative $\dot{\chi}_n(t)$, taken at any

[5] S. O. Rice, "Properties of a sine wave plus random noise," *Bell Sys. Tech. J.*, vol. 24, pp. 109–157; January, 1948. See p. 125.

[6] D. Middleton, "Spurious signals caused by noise in triggered circuits," *J. Appl. Phys.*, vol. 19, pp. 817–830; September, 1948. See p. 828.

[7] V. I. Bunimovich, "Amplitude peaks of random noise," *Zh. Tekh. Fiz.*, vol. 21, pp. 625–636; June, 1951. (In Russian.) See p. 630.

[8] See, for example, H. Cramér, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J., pp. 233–236; 1946.

[9] K. S. Miller and R. I. Bernstein, "Generalized Rayleigh processes" (abstract), *Bull. Amer. Math. Soc.*, vol. 63, pp. 196–197; May, 1957.

[10] K. S. Miller, R. I. Bernstein, and L. E. Blumenson, "Generalized Rayleigh processes," *Quart. Appl. Math.*, (to be published). These authors derive expressions for the bivariate and trivariate probability distributions of $\chi_n(t)$, and for its correlation function. In some cases they allow the components $g_i(t)$ to have means other than zero.

fixed t. We begin by writing the joint probability density of the $2n$ random variables $g_i(t)$, $\dot{g}_i(t)$, $1 \leq i \leq n$, *i.e.*, the values of the components of $\chi_n(t)$ and their derivatives, taken at any fixed t. It follows by the independence of the processes $g_i(t)$, $1 \leq i \leq n$, that

$$\begin{aligned}\langle g_i(t) g_j(t)\rangle &= \langle g_i(t)\dot{g}_j(t)\rangle \\ &= \langle \dot{g}_i(t)\dot{g}_j(t)\rangle = 0, \qquad 1 \leq i \neq j \leq n,\end{aligned} \tag{11}$$

and by differentiating (7) that

$$\langle g_i(t)\dot{g}_i(t)\rangle = \sigma^2 R'(0) = 0,$$

where $R'(0)$ vanishes by the evenness of the correlation function $R(\tau)$. This means that the $2n$ jointly Gaussian random variables $g_i(t)$, $\dot{g}_i(t)$, $1 \leq i \leq n$ are uncorrelated and hence independent (being jointly Gaussian). Moreover, we have

$$\begin{aligned}\langle g_i(t)\rangle &= 0, \qquad \langle g_i^2(t)\rangle = \sigma^2 \qquad , \\ \langle \dot{g}_i(t)\rangle &= 0, \qquad \langle \dot{g}_i^2(t)\rangle = -\sigma^2 R_0'' > 0, \qquad 1 \leq i \leq n.\end{aligned}$$

Thus, the joint probability density of the $2n$ independent Gaussian random variables $g_i(t)$, $\dot{g}_i(t)$, $1 \leq i \leq n$ is[11]

$$p(g_1, \cdots, g_n, \dot{g}_1, \cdots, \dot{g}_n) = \frac{1}{(2\pi\sigma^2)^n(-R_0'')^{n/2}} \exp\left[-\frac{\sum_{i=1}^n g_i^2}{2\sigma^2} - \frac{\sum_{i=1}^n \dot{g}_i^2}{-2\sigma^2 R_0''}\right],$$

where for notational simplicity the superfluous argument t has been omitted.

We now introduce n-dimensional spherical coordinates by writing

$$\begin{aligned}g_1 &= \chi_n \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1}, \\ g_2 &= \chi_n \sin\theta_1 \sin\theta_2 \cdots \cos\theta_{n-1}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ g_{n-1} &= \chi_n \sin\theta_1 \cos\theta_2 \qquad , \\ g_n &= \chi_n \cos\theta_1 \qquad ,\end{aligned} \tag{12}$$

where $\chi_n^2 = \sum_{i=1}^n g_i^2$. The transformation (12) maps the region $-\infty < g_i < \infty, 1 \leq i \leq n$ into the region

$$\begin{aligned}&0 \leq \chi_n < \infty, \\ &0 \leq \theta_1, \cdots, \theta_{n-2} \leq \pi, \\ &0 \leq \theta_{n-1} < 2\pi.\end{aligned}$$

Since

$$\begin{aligned}\sum_{i=1}^n \dot{g}_i^2 = \dot{\chi}_n^2 + \chi_n^2\dot{\theta}_1^2 + \chi_n^2\dot{\theta}_2^2 \sin^2\theta_1 + \cdots \\ + \chi_n^2\dot{\theta}_{n-1}^2 \sin^2\theta_1 \cdots \sin^2\theta_{n-2},\end{aligned} \tag{13}$$

the joint probability density of the new random variables $\chi_n, \theta_1, \cdots, \theta_{n-1}, \dot{\chi}_n, \dot{\theta}_1, \cdots, \dot{\theta}_{n-1}$ is easily seen to be

$$\begin{aligned}p(\chi_n, \theta_1, \cdots, \theta_{n-1}, \dot{\chi}_n, \dot{\theta}_1, \cdots, \dot{\theta}_{n-1}) \\ = \frac{1}{(2\pi\sigma^2)^n(-R_0'')^{n/2}}\exp\Bigg[-\frac{\chi_n^2}{2\sigma^2} \\ - \frac{\dot{\chi}_n^2 + \chi_n^2\dot{\theta}_1^2 + \chi_n^2\dot{\theta}_2^2\sin^2\theta_1 + \cdots + \chi_n^2\dot{\theta}_{n-1}^2\sin^2\theta_1\cdots\sin^2\theta_{n-2}}{-2\sigma^2R_0''}\Bigg] \\ \times \frac{\partial(g_1, g_2, \cdots, g_n, \dot{g}_1, \dot{g}_2, \cdots, \dot{g}_n)}{\partial(\chi_n, \theta_1, \cdots, \theta_{n-1}, \dot{\chi}_n, \dot{\theta}_1, \cdots, \dot{\theta}_{n-1})}.\end{aligned} \tag{14}$$

Since the partial derivatives of the variables $g_1, g_2, \cdots, g_n$ with respect to the variables $\dot{\chi}_n, \dot{\theta}_1, \cdots, \dot{\theta}_{n-1}$ vanish, the Jacobian determinant in (14) reduces to

$$\frac{\partial(g_1, g_2, \cdots, g_n)}{\partial(\chi_n, \theta_1, \cdots, \theta_{n-1})}\frac{\partial(\dot{g}_1, \dot{g}_2, \cdots, \dot{g}_n)}{\partial(\dot{\chi}_n, \dot{\theta}_1, \cdots, \dot{\theta}_{n-1})} = \left[\frac{\partial(g_1, g_2, \cdots, g_n)}{\partial(\chi_n, \theta_1, \cdots, \theta_{n-1})}\right]^2 = J_n^2,$$

where the new Jacobian

$$J_n = \chi_n^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}$$

is the square root of the product of the coefficients of the quantities $\dot{\chi}_n^2, \dot{\theta}_1^2, \cdots, \dot{\theta}_{n-1}^2$ in the sum (13). It follows by integrating (14) with respect to the angle variables and their time derivatives that the bivariate probability density of χ_n and $\dot{\chi}_n$ is given by

$$\begin{aligned}p(\chi_n, \dot{\chi}_n) = \frac{1}{(2\pi\sigma^2)^n(-R_0'')^{n/2}}\int_0^\pi d\theta_1 \\ \cdots \int_0^\pi d\theta_{n-2}\int_0^{2\pi} d\theta_{n-1} J_n^2 \exp\left[-\frac{\chi_n^2}{2\sigma^2} - \frac{\dot{\chi}_n^2}{-2\sigma^2R_0''}\right] \\ \cdot\int_{-\infty}^{\infty} d\dot{\theta}_1 \exp\left[-\frac{\chi_n^2\dot{\theta}_1^2}{-2\sigma^2R_0''}\right]\int_{-\infty}^{\infty} d\dot{\theta}_2 \exp\left[-\frac{\chi_n^2\dot{\theta}_2^2\sin^2\theta_1}{-2\sigma^2R_0''}\right] \\ \cdots\int_{-\infty}^{\infty} d\dot{\theta}_{n-1}\exp\left[-\frac{\chi_n^2\dot{\theta}_{n-1}^2\sin^2\theta_1\cdots\sin^2\theta_{n-2}}{-2\sigma^2R_0''}\right].\end{aligned} \tag{15}$$

The integrals with respect to $\dot{\theta}_1, \cdots, \dot{\theta}_{n-1}$ can be done immediately; their product is

$$\left[\frac{-2\pi\sigma^2R_0''}{\chi_n^2}\right]^{(n-1)/2}\frac{1}{\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}} = \frac{(-2\pi\sigma^2R_0'')^{(n-1)/2}}{J_n}. \tag{16}$$

Moreover, we have

$$\chi_n^{1-n}\int_0^\pi d\theta_1\cdots\int_0^\pi d\theta_{n-2}\int_0^{2\pi} d\theta_{n-1}J_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}, \tag{17}$$

i.e., the surface area of the n-dimensional sphere with unit radius. Thus we find, using (9), (16), and (17), that (15) reduces to

$$p(\chi_n, \dot{\chi}_n) = W_n(\chi_n)\frac{1}{\sqrt{-2\pi R_0''}\,\sigma}\exp\left[-\frac{\dot{\chi}_n^2}{-2\sigma^2R_0''}\right]. \tag{18}$$

It follows from (18) that at any time t, the process $\chi_n(t)$ and its derivative $\dot{\chi}_n(t)$ are independent. Indeed, the

[11] In the remainder of this section we shall use the same symbol to denote a random variable and the dummy variable corresponding to it in probability density functions. This slight abuse of notation allows us to avoid the introduction of numerous additional symbols.

nivariate distribution of $\dot{\chi}_n(t)$ is Gaussian with mean ero and variance $-\sigma^2 R_0''$, although its integral $\chi_n(t)$ s manifestly non-Gaussian. This involves no contradiction ince $\dot{\chi}_n(t)$ is not a Gaussian *process*, *i.e.*, the joint distri-ution of its values taken at different times is not multi-ariate Gaussian.

The proof of the theorem is completed by using (1) nd (18), whence

$$
\begin{aligned}
N_n(\alpha) &= \int_{-\infty}^{\infty} |\dot{\chi}_n| \, p(\alpha, \dot{\chi}_n) \, d\dot{\chi}_n \\
&= \frac{2W_n(\alpha)}{\sqrt{-2\pi R_0''}\,\sigma} \int_0^{\infty} \dot{\chi}_n \exp\left[-\frac{\dot{\chi}_n^2}{-2\sigma^2 R_0''}\right] d\dot{\chi}_n,
\end{aligned}
$$

o that

$$
N_n(\alpha) = \sqrt{\frac{2}{\pi}}\,\sigma\sqrt{-R_0''}\,W_n(\alpha), \qquad \text{Q.E.D.} \tag{10}
$$

In proving the theorem, the independence of the rocesses $g_1(t), \cdots, g_n(t)$ was used only to deduce the elations (11). Thus we have the following result.

orollary

Let $g_1(t), \cdots, g_n(t)$ be n stationary and jointly Gaussian andom processes which have zero means and satisfy (7). *uppose in addition that the processes $g_1(t), \cdots, g_n(t)$ are ot independent but are stationarily correlated, and that the oss correlations of $g_1(t), \cdots, g_n(t)$ satisfy* (11). *Then the xpected number of times per second that the process*

$$
\hat{\chi}_n(t) = \left[\sum_{i=1}^{n} g_i^2(t)\right]^{1/2} \tag{19}
$$

ssumes the value α is given by (10).

The difference between the process $\hat{\chi}_n(t)$ and the chi rocess $\chi_n(t)$ is, of course, that the components $g_i(t)$ guring in the definition (19) are *not* independent. How-ver, according to the corollary, which will be needed in he next section, the fluctuation rate of $\hat{\chi}_n(t)$ is also given y (10), provided that *simultaneous* values of the com-onents of $\hat{\chi}_n(t)$ and their derivatives are uncorrelated and hence independent since the $g_i(t)$ are jointly Gaussian rocesses). On the other hand, independence of the omponents would imply the much stronger condition

$$
\langle g_i(t) g_j(t')\rangle = \langle g_i(t)\dot{g}_j(t')\rangle = \langle \dot{g}_i(t)\dot{g}_j(t')\rangle = 0, \qquad 1 \le i \ne j \le n,
$$

or *all* t and t', not just for $t = t'$.

Application to the Envelope Fluctuation Rate

We shall now show that the general expression (10) ncludes as a special case the formula (4) for the fluctuation ate of the envelope $A(t)$ of a zero-mean stationary aussian process $g(t)$. First we recall the following facts bout the envelope.[12,13]

1) For every $\omega_0 > 0$, there is a representation of $g(t)$ of the form

$$
g(t) = g_c(t)\cos\omega_0 t - g_s(t)\sin\omega_0 t,
$$

where the processes $g_c(t)$ and $g_s(t)$ are jointly Gaussian, and

$$
\begin{aligned}
\langle g_c(t)\rangle &= \langle g_s(t)\rangle = \langle g(t)\rangle = 0, \\
\langle g_c^2(t)\rangle &= \langle g_s^2(t)\rangle = \langle g^2(t)\rangle = \sigma^2.
\end{aligned}
$$

2) The correlation properties of $g_c(t)$ and $g_s(t)$ are given by

$$
\begin{aligned}
\langle g_c(t) g_c(t+\tau)\rangle &= \langle g_s(t) g_s(t+\tau)\rangle = \sigma^2 R(\tau), \\
\langle g_s(t) g_s(t+\tau)\rangle &= -\langle g_s(t) g_c(t+\tau)\rangle = \sigma^2 S(\tau),
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
R(\tau) &= \int_0^{\infty} \Phi(\omega)\cos[(\omega-\omega_0)\tau]\,d\omega, \\
S(\tau) &= \int_0^{\infty} \Phi(\omega)\sin[(\omega-\omega_0)\tau]\,d\omega,
\end{aligned} \tag{21}
$$

and $\Phi(\omega)$ is the power spectrum of the underlying Gaussian process $g(t)$. The quantity

$$
k^2(\tau) = R^2(\tau) + S^2(\tau) \tag{22}
$$

is independent of ω_0. It is important to note that the processes $g_c(t)$ and $g_s(t)$ are independent only when $S(\tau)$ vanishes, as it does, for example, when $\Phi(\omega)$ is symmetric about ω_0.

3) In particular, it follows from (20) and (21) that[14]

$$
\begin{aligned}
\langle g_c(t) g_s(t)\rangle &= S_0 = 0, \\
\langle \dot{g}_c(t)\dot{g}_s(t)\rangle &= -S_0'' = 0,
\end{aligned} \tag{23}
$$

and that

$$
\begin{aligned}
\langle g_c(t)\dot{g}_s(t)\rangle &= -\langle \dot{g}_c(t) g_s(t)\rangle = S_0' \\
&= \int_0^{\infty} (\omega-\omega_0)\Phi(\omega)\,d\omega = \bar{\omega} - \omega_0,
\end{aligned} \tag{24}
$$

where $\bar{\omega}$ is the mean frequency of $\Phi(\omega)$.

4) The envelope $A(t)$ is defined by

$$
A(t) = [g_c^2(t) + g_s^2(t)]^{1/2},
$$

and is independent of the choice of ω_0.

The process $A(t)$ is not in general a chi process since the component processes $g_c(t)$ and $g_s(t)$ are in general dependent. However, it follows from (23) and (24) that the conditions needed to apply the corollary of the fluctuation rate theorem are met, provided that we choose the frequency ω_0, which is at our disposal, to be the mean frequency $\bar{\omega}$, so that S_0' vanishes. With this choice

[12] Rice, footnote 1, pp. 81–85.

[13] V. I. Bunimovich, "The fluctuating process as an oscillation ith random amplitude and phase," *Zh. Tekh. Fiz.*, vol. 19, pp. 231–1259; November, 1949. (In Russian.)

[14] The subscript zero denotes evaluation at $\tau = 0$.

$$-R_0'' = \int_0^\infty (\omega - \bar{\omega})^2 \Phi(\omega)\, d\omega = (\Delta\omega)^2,$$

where $(\Delta\omega)^2$ is the quantity defined by (6). Thus the envelope fluctuation rate $N_A(\alpha)$ is given by

$$N_A(\alpha) = N_2(\alpha) = \sqrt{\frac{2}{\pi}}\, \sigma \sqrt{-R_0''}\, W_2(\alpha),$$

or

$$N_A(\alpha) = \sqrt{\frac{2}{\pi}}\, \sigma\, \Delta\omega\, W_2(\alpha), \tag{4}$$

which completes the proof that (4) is a special case of (10).

It is interesting to note another expression for $N_A(\alpha)$ in terms of the *envelope correlation function*

$$R_A(\tau) = \frac{\langle A(t)A(t+\tau)\rangle - \langle A(t)\rangle^2}{\langle A^2(t)\rangle - \langle A(t)\rangle^2}.$$

It can be shown that[15]

$$R_A(\tau) = \frac{2E(k) - (1 - k^2)K(k) - (\pi/2)}{2 - (\pi/2)},$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively, and k is the function of τ defined by (22). It follows that

$$-R_A''(0) = \frac{2}{4 - \pi} \frac{\partial}{\partial k} [2E(k) - (1 - k^2)K(k)]_{k=1} (-R_0'' - S_0'^2) = \frac{2}{4 - \pi} (\Delta\omega)^2, \tag{25}$$

where we have used formulas 112.01, 710.02, and 710.04 from Byrd and Friedman's handbook of elliptic integrals.[16] Comparing (4) and (25), we see that another expression for $N_A(\alpha)$ is

$$N_A(\alpha) = \sqrt{\frac{4 - \pi}{\pi}}\, \sigma \sqrt{-R_A''(0)}\, W_2(\alpha).$$

[15] See, for example, Bunimovich, footnote 13, p. 1248.

[16] P. F. Byrd and M. D. Friedman, "Handbook of Elliptic Integrals for Engineers and Physicists," Lange, Maxwell, and Springer, Ltd., London, Eng., and New York, N. Y.; 1954.

Loss of Signal Detectability in Band-Pass Limiters*

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Summary—A loss of signal detectability results from ideal symmetric limiting of a very narrow-band signal in narrow-band Gaussian noise. At small input snr this loss can be expressed in terms of the degradation of the effective signal energy-to-noise power per cycle ratio. Proceeding from results derived previously by Davenport and Price, an expression is derived for this degradation in terms of the autocorrelation function of the input noise. The degradation is evaluated for three typical input noise autocorrelation functions and is found to be quite small in all these cases. It is seen that the degradation can be made to vanish by appropriately shaping the spectrum of the input noise. Regulation of the input noise spectrum to obtain conditions of limiter operation assumed in this paper may often prove to be a convenient method for reducing loss of signal detectability in band-pass limiters.

THE effect of nonlinear devices on a signal imbedded in noise is frequently analyzed in terms of a signal-to-noise (power) ratio as the significant parameter. Where fidelity of transmission is of concern the snr description has, of course, much to recommend it. However, when the presence of the signal is not a foregone conclusion and signal detectability is of primary concern, the snr description is no longer appropriate.

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It has been shown that the effective detectability of an exactly known signal in the presence of additive white Gaussian noise is characterized by E/N_0, the ratio of the signal energy to the noise power per unit bandwidth.[1] Any irreversible operation (squaring, rectification, envelope detection, limiting) performed on the received message waveform usually destroys information and tends to decrease the detectability of the signal. This inherent loss in signal detectability can be characterized by a decrease in the effective value of E/N_0 provided that:

1) The input signal-to-noise (power) ratio is small (so that output spectrum is virtually the same whether or not the signal is present).

2) The noise resulting from the nonlinear operation is equivalent to white Gaussian noise over the region of interest.

3) The resulting signal remains well defined.

The important irreversible operation to be considered here is the symmetrical ideal band-pass limiter operated at low input snr (see Fig. 1).

[1] W. W. Peterson, T. G. Birdsall, and W. C. Fox, "The theory of signal detectability," IRE Trans. on Information Theory, vol. 4, pp. 171–212; September, 1954.

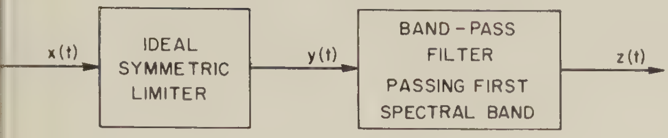

Fig. 1—Band-pass limiter.

The output of the limiter, $y(t)$, is the following instan-aneous nonlinear function of the input, $x(t)$.

$$y(t) = \begin{Bmatrix} +1 & \text{if} & x(t) > 0 \\ 0 & \text{if} & x(t) = 0 \\ -1 & \text{if} & x(t) < 0 \end{Bmatrix}. \qquad (1)$$

he band-pass filter is designed to pass, without dis-ortion, only the spectral band of $y(t)$ near the frequency f the input. Spectral bands at harmonics of the input requency are rejected by this filter. The output of this and-pass filter is denoted by $z(t)$.

Davenport[2] has analyzed the effect of the band-pass miter on snr for a sine wave signal imbedded in narrow-and Gaussian noise. In particular, Davenport finds hat when the input snr, $(S/N)_{\text{in}}$, is much less than unity, he output signal is sinusoidal and the output snr, $S/N)_{\text{out}}$, is given by

$$(S/N)_{\text{out}} \equiv \frac{S_{\text{out}}}{N_{\text{out}}} = \frac{\pi}{4}\frac{S_{\text{in}}}{N_{\text{in}}} \equiv \frac{\pi}{4}(S/N)_{\text{in}} \qquad (2)$$

here S_{in}, S_{out}, N_{in}, N_{out} are the respective signal and oise powers at the input and output. It seems reasonable, hough rigorous justification is not given here, that Davenport's results hold also for the case of an arbitrary ignal imbedded in narrow-band Gaussian noise, provided hat the signal bandwidth is sufficiently small compared o the noise bandwidth. In particular it appears to be rue that the output signal is essentially an undistorted eplica of the input signal. We shall use these results in he discussion which follows.

The input waveform, $x(t)$, is the sum of a noise wave-orm, $n(t)$, and a signal waveform, $s(t)$, which may or nay not be present.

$$x(t) = \begin{Bmatrix} n(t) \\ \text{or} \\ s(t) + n(t) \end{Bmatrix}. \qquad (3)$$

he narrow-band Gaussian noise $n(t)$ centered in the icinity of the signal center frequency, denoted f_s, has (two-sided) power spectrum depending on frequency f, enoted $N_{\text{in}}(f)$. $s(t)$ is assumed to be an exactly known, ery narrow-band signal with energy E_{in} given by

$$E_{\text{in}} = \int_{-\infty}^{\infty} s(t)^2\,dt. \qquad (4)$$

he corresponding signal energy component of $z(t)$ is enoted E_{out} while the corresponding noise power spectrum component of $z(t)$ is denoted $N_{\text{out}}(f)$. $N_{\text{out}}(f)$ is approximately the same whether the signal is present or absent because of the assumption that $(S/N)_{\text{in}} \ll 1$.

If $N_{\text{in}}(f)$ is approximately constant over the frequency band of $s(t)$ so that it may be considered to constitute a white Gaussian noise background, the inherent detectability of the signal at the input is characterized by[3]

$$(E/N_0)_{\text{in}} = \frac{E_{\text{in}}}{2N_{\text{in}}(f_s)}. \qquad (5)$$

If $N_{\text{out}}(f)$ is approximately constant over the frequency band of $s(t)$ and if the bandwidth of $s(t)$ is sufficiently narrow compared to the bandwidth of $n(t)$, it appears, by virtue of the central limit theorem, that $N_{\text{out}}(f)$ is equivalent to a white Gaussian noise background over the bandwidth of the signal.[4] Therefore, the inherent detectability of the signal, given $z(t)$, is characterized by

$$(E/N_0)_{\text{out}} = \frac{E_{\text{out}}}{2N_{\text{out}}(f_s)}. \qquad (6)$$

The degradation factor, denoted Λ, which represents the inherent loss of signal detectability through the band-pass limiter is then given by

$$\Lambda = \frac{(E/N_0)_{\text{in}}}{(E/N_0)_{\text{out}}} = \frac{E_{\text{in}}N_{\text{out}}(f_s)}{E_{\text{out}}N_{\text{in}}(f_s)}. \qquad (7)$$

Noting from (2) that

$$\frac{E_{\text{in}}}{E_{\text{out}}} = \frac{S_{\text{in}}}{S_{\text{out}}} = \frac{(S/N)_{\text{in}}N_{\text{in}}}{(S/N)_{\text{out}}N_{\text{out}}} = \frac{4N_{\text{in}}}{\pi N_{\text{out}}} \qquad (8)$$

we have for the degradation factor

$$\Lambda = \frac{4}{\pi}\frac{N_{\text{in}}}{N_{\text{in}}(f_s)}\frac{N_{\text{out}}(f_s)}{N_{\text{out}}}. \qquad (9)$$

The main object of this paper is to evaluate Λ and its dependence on the shape of $N_{\text{in}}(f)$.

The band-pass limiter shown in Fig. 1 may be thought of as a device in which the limiter portion strips the amplitude variations from its input, while the subsequent filter, centered on the input frequency, is just broad enough to pass the input phase variations without distortion. Thus, at low input snr, the output noise is essentially equal to the phase modulated component of the input noise. Following the terminology of Price,[5] the correlation function of the input noise is written

$$\varphi_n(\tau) = \overline{n(t)n(t+\tau)} = \varphi_n(0)\sigma_\tau \cos\left[2\pi f_n \tau + \lambda(\tau)\right] \qquad (10)$$

where we identify $\varphi_n(0) = N_{\text{in}}$. f_n is the (arbitrarily

[2] W. B. Davenport, Jr., "Signal-to-noise ratios in band-pass miters," *J. Appl. Phys.*, vol. 24, pp. 720–727; June, 1953.

[3] The optimum detectability can be obtained with the aid of a correlation or matched-filter detector operating on $x(t)$. These detectors achieve an effective snr equal to $(2E/N_0)_{\text{in}}$. For further details see Peterson, *et al.*, *op. cit.*

[4] We are assuming here that the band-pass limiter is followed by a filter which is matched to $s(t)$.

[5] R. Price, "A note on the envelope and phase-modulated components of narrow-band Gaussian noise," IRE Trans. on Information Theory, vol. 1, pp. 9–13; September, 1955. Bunimovich obtained (5a) and (6) of this paper in 1949.

V. I. Bunimovich, "Fluctuation processes as oscillations of random amplitude and phase," *Zh. Tekh. Fiz.* (*Leningrad*), vol. 19, pp. 1231–1259; November, 1949. See (70) and (70a).

chosen) center frequency of $N_{in}(f)$. σ_τ represents the normalized amplitude of $\varphi_n(\tau)$ and satisfies $\sigma_0 = 1$. $\lambda(\tau)$ is a phase modulating term which satisfies $\lambda(-\tau) = -\lambda(\tau)$ and vanishes when $N_{in}(f)$ is locally symmetric about f_n. From (5a), (5b), and (6) of the Price paper[5] we see that the autocorrelation function of the output, ignoring the signal component, is given by

$$\varphi_z(\tau) = \overline{z(t)z(t+\tau)}$$

$$= \frac{\pi\sigma_\tau}{8} \cos\left[2\pi f_n\tau + \lambda(\tau)\right] \cdot \left\{1 + \sum_{k=1}^{\infty} \frac{\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k-1}{2}\right)\right]^2}{k!(k+1)!}\sigma_\tau^{2k}\right\} \quad (11)$$

where we identify $\varphi_z(0) = N_{out} = 1/2$. Using (10) and (11), introducing the necessary Fourier transformations and substituting in (9) we have

$$\Lambda = \frac{\int_0^\infty \sigma_\tau \cos\left[2\pi f_n\tau + \lambda(\tau)\right]\left\{1 + \sum_{k=1}^{\infty} \frac{\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k-1}{2}\right)\right]^2}{k!(k+1)!}\sigma_\tau^{2k}\right\} \cos 2\pi f_s\tau \, d\tau}{\int_0^\infty \sigma_\tau \cos\left[2\pi f_n\tau + \lambda(\tau)\right] \cos 2\pi f_s\tau \, d\tau}. \quad (12)$$

It is easily seen from the above equation that Λ is invariant to a change in τ scale and hence to a change in the spectral frequency scale.

It is apparent from (12) that the degradation in signal detectability comes about from the integral terms in the numerator involving σ_τ^{2k}. That each of these terms is positive can be seen by converting the Fourier transform to a multiple convolution of nonnegative spectral densities. Although σ_τ cannot in general represent a true autocorrelation function having a nonnegative spectral density, σ_τ^2 must always be an autocorrelation function. This result is shown by taking two independent band-pass noises each with autocorrelation function $\varphi_n(\tau)$. Multiplying them, we obtain a process whose autocorrelation function is $\varphi_n^2(\tau)$ and whose power spectrum in the region around dc is seen to be the transform of $\frac{1}{2}\varphi_n^2(0)\,\sigma_\tau^2$, using (10).

Often when limiting action is introduced, purposely or otherwise, into a receiver, the shape of $N_{in}(f)$ can be regulated to some extent. The minimization of Λ with respect to $N_{in}(f)$, subject to reasonable physical constraints on $N_{in}(f)$, does not appear to be capable of solution by the calculus of variations. However, consider the spectrum shape for $N_{in}(f)$ shown in Fig. 2. The total input noise spectrum is the sum of two parts, A and B. Region B is wide enough to include the signal, and the height of A is negligible compared to that of B, but the power in B is negligible compared to that of A. With such a spectrum terms involving higher powers of σ_τ in (12) are negligible and $\Lambda = 1$. This result is somewhat surprising, especially since it can be interpreted to mean that adding noise to the input, *outside* the signal band will actually yield a reduction in detectability loss. The proof that Λ approaches unity in the limit is given in the Appendix.

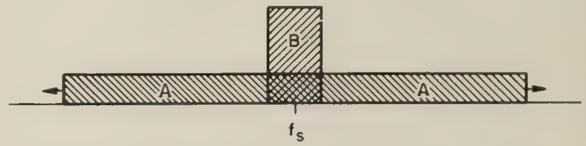

Fig. 2—An optimum noise spectrum.

It is of interest to find the value of Λ for several typical shapes of $N_{in}(f)$. For these cases we assume that $N_{in}(f)$ is locally symmetric about frequency f_n so that $\lambda(\tau) = 0$. It is further assumed that $f_s = f_n$. Then, since σ_τ is slowly varying with respect to $\cos 4\pi f_n\tau$, (12) reduces quite accurately to

$$\Lambda = 1 + \frac{\sum_{k=1}^{\infty} \frac{\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k-1}{2}\right)\right]^2}{k!(k+1)!} \int_0^\infty \sigma_\tau^{2k+1}\, d\tau}{\int_0^\infty \sigma_\tau\, d\tau}$$

$$= 1 + \frac{1}{8}\frac{\int_0^\infty \sigma_\tau^3\, d\tau}{\int_0^\infty \sigma_\tau\, d\tau} + \frac{3}{64}\frac{\int_0^\infty \sigma_\tau^5\, d\tau}{\int_0^\infty \sigma_\tau\, d\tau} + \frac{25}{1024}\frac{\int_0^\infty \sigma_\tau^7\, d\tau}{\int_0^\infty \sigma_\tau\, d\tau} + \cdots. \quad (13)$$

Using this expression the following results are obtained:

1) Rectangular $N_{in}(f)$: For a rectangular $N_{in}(f)$ which we take to be of width $1/\pi$ since scale does not matter, σ_τ has the form

$$\sigma_\tau = \frac{\sin\tau}{\tau}. \quad (14)$$

Making use of the following integral,[6]

$$\int_0^\infty \left(\frac{\sin\tau}{\tau}\right)^{2k+1} d\tau = \frac{\pi}{2^{2k+1}(2k)!}\sum_{j=0}^{k}\binom{2k+1}{j}(-1)^j(2k-2j+1)^{2k} \qquad k \geq 0 \quad (15)$$

[6] Bateman Manuscript Project, "Tables of Integral Transforms," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 1, p. 20; 1954. See formula 11.

here $\binom{2k+1}{j}$ is a binomial coefficient, we obtain for Λ

$$\Lambda = 1 + \sum_{k=1}^{\infty} \frac{\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k-1}{2}\right)\right]^2}{k!(k+1)!(2k)!2^{2k}} \cdot \sum_{j=0}^{k} \binom{2k+1}{j}(-1)^j(2k-2j+1)^{2k}$$
$$\approx 1.16. \tag{16}$$

When the signal is not centered in the middle of the ectangular band of input noise, Λ can be computed by esorting to (9), where $N_{\text{out}}(f_s)$ is available from numerical 'ourier transforms of (11) performed by Price.[5] The esults show that Λ decreases slowly as f_s approaches the and edge and is approximately 1.12 just short of the and edge.

2) Gaussian shaped $N_{\text{in}}(f)$: Again choosing a convenient cale factor, σ_τ has the form

$$\sigma_\tau = e^{-\tau^2} \tag{17}$$

iving for Λ

$$\Lambda = 1 + \sum_{k=1}^{\infty} \frac{\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k-1}{2}\right)\right]^2}{k!(k+1)!\sqrt{2k+1}} \approx 1.118. \tag{18}$$

3) Optically shaped $N_{\text{in}}(f)$: Again choosing a con-enient scale factor, σ_τ has the form

$$\sigma_\tau = e^{-|\tau|} \tag{19}$$

iving for Λ

$$\Lambda = 1 + \sum_{k=1}^{\infty} \frac{\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k-1}{2}\right)\right]^2}{k!(k+1)!(2k+1)} \approx 1.059. \tag{20}$$

'his last case has the smallest degradation presumably ecause the power spectrum $N_{\text{in}}(f)$ comes closest in some ense to approximating the shape shown in Fig. 2.

The most interesting and important conclusion to be oted from these results is that the degradation in de-ectability is quite small in all the cases considered. The esults suggest that appropriate regulation of $N_{\text{in}}(f)$ to btain conditions of band-pass limiter operation assumed arlier, *viz.*, $(S/N)_{\text{in}} \ll 1$ and signal bandwidth much maller than input noise bandwidth, may often prove to e a convenient means for reducing loss in signal de-ectability brought about by the limiter.

In situations where detectability is not of primary oncern snr is sometimes used to characterize the per-ormance of a limiter. Curves given by Davenport[2] for he band-pass limiter show that a two-fold increase in nr results when the limiter is operated at high input nr. One should not conclude from this result that the imiter should be operated at high input snr for best etection performance. As we have seen, it is possible to perate the limiter so that the inherent detectability of he signal is essentially unchanged even though the snr ecreases. SNR for a finite duration signal may sometimes represent a lower bound on signal detectability, but this quantity says nothing about the inherent detectability which can be obtained with further filtering and consequent improvement in snr.

The increase in snr obtained with the band-pass limiter is very similar to the increase in snr obtained from the band-pass envelope detector operated at high input snr[7] (certainly a device which usually degrades signal detectability). A factor of two increase in snr results in each case because the envelope detector suppresses the component of noise out of phase with the signal while the limiter suppresses the component of noise in phase with the signal.

Appendix

It is desired to prove that with the spectrum shape for $N_{\text{in}}(f)$ shown in Fig. 2 all terms in the numerator of (12) involving σ_τ^{2k} are negligible (so that Λ approaches unity) in the limit where the height of A is negligible compared to that of B, but the power in B is negligible compared to that of A.

We denote the respective widths of regions A and B in Fig. 2 by W_A and W_B and their respective heights by N_A and N_B. Eq. (12) can be written

$$\Lambda - 1 = \frac{\sum_{k=1}^{\infty} \frac{\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k-1}{2}\right)\right]^2}{k!(k+1)!} \int_0^{\infty} \varphi_n(\tau)\sigma_\tau^{2k} \cos 2\pi f_s\tau\, d\tau}{\int_0^{\infty} \varphi_n(\tau) \cos 2\pi f_s \tau\, d\tau}. \tag{21}$$

The denominator on the right will be recognized as simply $\frac{1}{2} N_{\text{in}}(f_s)$. Since f_s is always taken to lie in the region B, we have $N_{\text{in}}(f_s) \geqq N_B$. The series in the numerator above is dominated, term by term, by the following series

$$\sum_{k=1}^{\infty} \frac{\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2k-1}{2}\right)\right]^2}{k!(k+1)!} \int_{-\infty}^{\infty} \varphi_n(0)\sigma_\tau^2\, d\tau \tag{22}$$

where the integral, being independent of k, may be factored out of the sum. The sum converges to a finite positive constant [which can be evaluated by setting $\tau = 0$, $\varphi_z(0) = \frac{1}{2}$ in (11)]. The σ_τ and $\lambda(\tau)$ corresponding to a very narrow-band noise input are slowly varying compared to cos $(2\pi f_n \tau)$. Therefore

$$\int_{-\infty}^{\infty} \varphi_n^2(\tau)\, d\tau = \tfrac{1}{2}\varphi_n^2(0) \int_{-\infty}^{\infty} \sigma_\tau^2\, d\tau. \tag{23}$$

From the above equations it follows that

$$\Lambda - 1 \leqq \frac{c \int_{-\infty}^{\infty} \varphi_n^2(\tau)\, d\tau}{N_B \varphi_n(0)} \tag{24}$$

where c represents a positive constant. By Parseval's theorem, the integral in (24) can be converted to an

[7] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 24, pp. 46–156; January, 1945. See sec. 3.10.

integral over $N_{in}^2(f)$, and this yields

$$\int_{-\infty}^{\infty} \varphi_n^2(\tau)\, d\tau = 2(W_A - W_B)N_A^2 + 2W_B(N_A + N_B)^2. \quad (25)$$

Note also

$$\varphi_n(0) = 2W_AN_A + 2W_BN_B. \quad (26)$$

Substituting (25) and (26) into (24), we have

$$\Lambda - 1 \leqq c\,\frac{[(W_A - W_B)N_A^2 + W_B(N_A + N_B)^2]}{N_B(W_AN_A + W_BN_B)}$$

$$\leqq c\left[\frac{N_A}{N_B} + \frac{W_BN_B}{W_AN_A} + 2\,\frac{W_B}{W_A}\right]. \quad (27)$$

In order for the right-hand side to approach zero it is necessary and sufficient that

$$\frac{N_A}{N_B} \to 0 \quad \text{and} \quad \frac{W_BN_B}{W_AN_A} \to 0 \quad (28)$$

thus demonstrating the requirements on $N_{in}(f)$ stated earlier. However, these requirements can be met only insofar as bandwidth limitations will allow.

Note that nothing in our analysis has precluded the possibility of region A in Fig. 2 not overlapping region B containing the signal. This suggests the interesting possibility of using noise outside the band B of signal and noise both to control the limiting process and to reduce the loss in signal detectability.

A Table of Bias Levels Useful in Radar Detection Problems*

JAMES PACHARES†

Summary—A table of bias levels $\lambda = \lambda(m,p)$ corresponding to specified probabilities of false alarm, 10^{-p}, when integrating over m independent noise pulses using a radar system with a square-law detector receiver, is given for $p = 1(1)12$ and $m = 1(1)150$. The notation $p = 1(1)12$ means that p varies from 1 to 12 in steps of unity.

Introduction

Assume that noise pulses result in signals whose amplitudes, r, have the probability density function given by

$$e^{-r^2/2\sigma^2}(r/\sigma^2). \quad (1)$$

Then, if R_m^2 represents the sum of squares of the amplitudes of m independent such pulses, it is well known that $t = R_m^2/2\sigma^2$ has the probability density function given by

$$e^{-t}t^{m-1}/(m-1)!. \quad (2)$$

As a result of (1) the average value of r^2 is $2\sigma^2$ and the average value of r is $\sigma\sqrt{\pi/2}$.

If the bias level is set at c, then the probability of a false alarm equals the probability that R_m^2 exceeds c, which equals the probability that t exceeds $c/2\sigma^2$.

If 10^{-p} is the probability of false alarm when integrating over m independent noise pulses using a square-law detector, then p, m, and λ are related by the following expression:

$$\int_{\lambda}^{\infty} \frac{e^{-t}t^{m-1}}{(m-1)!}\, dt = 10^{-p} \quad (3)$$

where $c = 2\sigma^2\lambda$.[1]

In practice one usually has a value of m and p in mind and would like to know the corresponding value of the bias level, λ.

Example: Suppose one wants the value of c such that the probability of a false alarm will be one chance in a million when integrating over 25 independent noise pulses when $\sigma^2 = 2$. From the tables we find that for $m = 25$ and $p = 6$, $\lambda = 56.3040$. Consequently, $c = 2\sigma^2\lambda = 225.2160$.

Tables of the bias level $\lambda = \lambda\,(m, p)$ are given correctly to four decimal places for $p = 1(1)12$ and $m = 1(1)150$. This table adequately covers the range which is likely to be of interest in the above type situations. For $m > 150$ one could use the asymptotic formula given by Wishart[2] or the asymptotic formula (7), to find λ. It should be noted that since the values of λ given in the table are the probability points of the gamma distribution this table is of use in solving many statistical problems in which the gamma distribution is used.

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[1] J. I. Marcum, "A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix," Project Rand, RM-753, p. 11; April 25, 1952.

[2] J. Wishart, "χ^2 probabilities for large numbers of degrees of freedom," *Biometrica*, vol. 43, pts. 1–2, pp. 92–95; June, 1956.

These are the only known tables in existence which cover the above ranges in p. Pearson's tables[3] give the values corresponding to the integral (3) to seven decimals for m up to 51 so that one cannot find the bias level λ if $p \geq 7$. Pearson's tables not only do not cover the complete range of interest but also, in situations where a point of interest does fall within the range of the table, *i.e.*, if $m \leq 51$ and $p \leq 6$, one would have to undergo the inconvenience of using inverse interpolation to find λ.

Tables have been computed by Teichroew[4] giving λ for $m = 2(1)15, 20(10)50$, 100 and p up to 3 so that one could not find the bias level λ from these tables if $p > 3$.

Method Used to Compute the Tables

Newton's method was used to find the value of λ satisfying the equation $f(\lambda) = 0$, where

$$f(\lambda) = \int_\lambda^\infty \frac{e^{-t} t^{m-1}}{(m-1)!}\, dt - 10^{-p}.$$

If an initial guess, λ_0, say, is available, then as is well known, Newton's method yields a more accurate solution using

$$\lambda = \lambda_0 - f(\lambda_0)/f'(\lambda_0). \tag{4}$$

The following relation was used to find $f(\lambda)$:

$$\int_\lambda^\infty \frac{e^{-t} t^{m-1}}{(m-1)!}\, dt = \frac{e^{-\lambda}\lambda^{m-1}}{(m-1)!} S(m, \lambda) \tag{5}$$

where $S(m, \lambda)$ was computed from

$$S(m, \lambda) = 1 + \left(\frac{m-1}{\lambda}\right) \cdot\left\{1 + \cdots + \frac{3}{\lambda}\left[1 + \frac{2}{\lambda}\left(1 + \frac{1}{\lambda}\right)\right]\right\}$$

in order to preserve accuracy.

Since

$$f'(\lambda) = -\frac{e^{-\lambda}\lambda^{m-1}}{(m-1)!},$$

(4), using (5), becomes

$$\lambda = \lambda_0 + S(m, \lambda_0) - A(m, \lambda_0) \tag{6}$$

where $A(m, \lambda_0)$ was computed from

$$\log A(m, \lambda) = \lambda - (m-1)\log\lambda + \log(m-1)! - p\log 10.$$

The first guess for λ, λ_0, was found using

$$\lambda_0 = m + \sqrt{m}Q_1(x) + Q_2(x) + \frac{Q_3(x)}{\sqrt{m}} + \frac{Q_4(x)}{m} + \cdots + \frac{Q_8(x)}{m^3} \tag{7}$$

where the Q's are Campbell's polynomials given by

$$\begin{aligned}
Q_1(x) &= x \\
Q_2(x) &= (x^2 - 1)/3 \\
Q_3(x) &= (x^3 - 7x)/36 \\
Q_4(x) &= (-3x^4 - 7x^2 + 16)/810 \\
Q_5(x) &= (9x^5 + 256x^3 - 433x)/38880 \\
Q_6(x) &= (12x^6 - 243x^4 - 923x^2 + 1472)/204120 \\
Q_7(x) &= (-3753x^7 - 4353x^5 + 289517x^3 \\
&\qquad + 289717x)/146966400 \\
Q_8(x) &= (270x^8 + 4614x^6 - 9513x^4 \\
&\qquad - 104989x^2 + 35968)/55112400
\end{aligned}$$

and where x is found from

$$\int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt = 10^{-p},$$

using the literature.[5] Campbell[6] derived the asymptotic expansion (7) from the relation

$$\int_\lambda^\infty \frac{e^{-t} t^{m-1}}{(m-1)!}\, dt = \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt.$$

Only two iterations of Newton's method were required on the average.

Checking of Tabulated Values

Checks were made against the values given by Lewis[7] for $p = 3$ and $m = 20(5)50$, 60 and showed that our results were all good to at least four decimals. Checks made against the values given by Teichroew[4] for $p = 1$, 2, 3, and $m = 3, 50, 100$ again showed that our results were all good to at least four decimals. In addition, the table was differenced four times in both directions, in an effort to catch any obvious errors which might have occurred, with the result that no mistakes were apparent. Finally, several values were checked by hand computation to verify the results. Although λ was computed to seven decimals, only four decimals are given in the tables since the values were found, in the worst case, to be accurate to only four decimals.

Acknowledgment

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[3] K. Pearson, "Tables of the Incomplete Gamma Function," Cambridge Press, New York, N. Y., 1951.

[4] D. Teichroew, "A table of millile probability points of the incomplete gamma distribution," Natl. Bur. Standards (U. S.); February, 1954.

[5] "Tables of probability functions," prepared by U. S. Dept. of Commerce, Natl. Bur. Standards, vol. 2, 1948.

[6] G. A. Campbell, "Probability curves showing Poisson's Exponential summation," *Bell Sys. Tech. J.*, vol. 2, pp. 95–113; 1923.

[7] T. Lewis, "99.9 and 0.1% points of the χ^2 distribution," *Biometrika*, vol. 40, pts. 3–4, pp. 421–426; December, 1953.

Tables of $\lambda = \lambda(m, p)$

where

$$\int_{\lambda}^{\infty} \frac{e^{-t} t^{m-1}}{(m-1)!} dt = 10^{-p}$$

for $p = 1(1)12$ and $m = 1(1)150$

m \ p	1	2	3	4	5	6
1	2.3026	4.6052	6.9078	9.2103	11.5129	13.8155
2	3.8897	6.6384	9.2334	11.7564	14.2366	16.6884
3	5.3223	8.4059	11.2289	13.9282	16.5535	19.1292
4	6.6808	10.0451	13.0622	15.9138	18.6658	21.3505
5	7.9936	11.6046	14.7941	17.7820	20.6481	23.4315
6	9.2747	13.1085	16.4547	19.5672	22.5381	25.4126
7	10.5321	14.5706	18.0616	21.2896	24.3580	27.3177
8	11.7709	16.0000	19.6262	22.9624	26.1225	29.1622
9	12.9947	17.4027	21.1562	24.5947	27.8415	30.9571
10	14.2060	18.7831	22.6574	26.1930	29.5223	32.7103
11	15.4066	20.1447	24.1340	27.7623	31.1705	34.4279
12	16.5981	21.4899	25.5893	29.3065	32.7904	36.1144
13	17.7816	22.8208	27.0260	30.8286	34.3855	37.7737
14	18.9580	24.1391	28.4461	32.3312	35.9585	39.4087
15	20.1280	25.4461	29.8515	33.8163	37.5117	41.0221
16	21.2924	26.7429	31.2436	35.2856	39.0471	42.6158
17	22.4516	28.0305	32.6236	36.7406	40.5663	44.1916
18	23.6061	29.3096	33.9926	38.1825	42.0706	45.7512
19	24.7563	30.5810	35.3514	39.6124	43.5614	47.2958
20	25.9025	31.8454	36.7010	41.0311	45.0395	48.8265
21	27.0451	33.1031	38.0419	42.4397	46.5061	50.3444
22	28.1843	34.3548	39.3748	43.8386	47.9618	51.8503
23	29.3203	35.6007	40.7002	45.2287	49.4074	53.3451
24	30.4533	36.8413	42.0186	46.6105	50.8436	54.8295
25	31.5836	38.0769	43.3304	47.9844	52.2708	56.3040
26	32.7112	39.3079	44.6361	49.3509	53.6898	57.7694
27	33.8364	40.5344	45.9359	50.7106	55.1008	59.2260
28	34.9593	41.7567	47.2303	52.0636	56.5044	60.6744
29	36.0799	42.9751	48.5194	53.4105	57.9009	62.1150
30	37.1985	44.1897	49.8036	54.7515	59.2907	63.5482
31	38.3151	45.4008	51.0831	56.0868	60.6742	64.9743
32	39.4298	46.6084	52.3582	57.4169	62.0516	66.3937
33	40.5427	47.8129	53.6289	58.7418	63.4232	67.8068
34	41.6540	49.0142	54.8956	60.0619	64.7892	69.2137
35	42.7635	50.2126	56.1585	61.3773	66.1500	70.6147
36	43.8715	51.4082	57.4176	62.6883	67.5057	72.0102
37	44.9780	52.6010	58.6731	63.9950	68.8566	73.4002
38	46.0831	53.7913	59.9252	65.2977	70.2027	74.7851
39	47.1868	54.9790	61.1740	66.5964	71.5444	76.1651
40	48.2891	56.1644	62.4196	67.8913	72.8818	77.5403
41	49.3902	57.3474	63.6622	69.1825	74.2151	78.9108
42	50.4900	58.5283	64.9018	70.4703	75.5443	80.2770
43	51.5886	59.7069	66.1387	71.7546	76.8697	81.6388
44	52.6861	60.8836	67.3727	73.0357	78.1913	82.9965
45	53.7825	62.0582	68.6042	74.3136	79.5094	84.3503

m \ p	7	8	9	10	11	12
1	16.1181	18.4207	20.7233	23.0259	25.3284	27.6310
2	19.1198	21.5358	23.9397	26.3340	28.7203	31.0999
3	21.6689	24.1813	26.6723	29.1459	31.6052	34.0524
4	23.9862	26.5847	29.1538	31.6990	34.2244	36.7330

5	26.1548	28.8320	31.4727	34.0838	36.6702	39.2358
6	28.2168	30.9671	33.6746	36.3472	38.9907	41.6097
7	30.1976	33.0165	35.7867	38.5173	41.2148	43.8842
8	32.1137	34.9974	37.8271	40.6126	43.3613	46.0788
9	33.9766	36.9219	39.8083	42.6463	45.4439	48.2073
10	35.7947	38.7990	41.7396	44.6279	47.4725	50.2799
11	37.5744	40.6353	43.6280	46.5646	49.4544	52.3043
12	39.3207	42.4362	45.4790	48.4623	51.3957	54.2867
13	41.0377	44.2058	47.2972	50.3255	53.3012	56.2318
14	42.7286	45.9476	49.0859	52.1581	55.1746	58.1439
15	44.3960	47.6644	50.8484	53.9629	57.0193	60.0260
16	46.0422	49.3587	52.5870	55.7428	58.8379	61.8811
17	47.6692	51.0323	54.3038	57.4999	60.6327	63.7116
18	49.2785	52.6872	56.0008	59.2362	62.4058	65.5194
19	50.8717	54.3248	57.6795	60.9532	64.1588	67.3064
20	52.4498	55.9464	59.3412	62.6524	65.8933	69.0740
21	54.0141	57.5531	60.9873	64.3352	67.6105	70.8238
22	55.5654	59.1460	62.6187	66.0025	69.3116	72.5569
23	57.1047	60.7260	64.2365	67.6555	70.9978	74.2743
24	58.6326	62.2939	65.8414	69.2950	72.6698	75.9771
25	60.1499	63.8505	67.4343	70.9219	74.3286	77.6661
26	61.6572	65.3963	69.0158	72.5368	75.9749	79.3421
27	63.1551	66.9320	70.5867	74.1405	77.6094	81.0058
28	64.6441	68.4582	72.1474	75.7334	79.2328	82.6579
29	66.1246	69.9753	73.6984	77.3163	80.8455	84.2989
30	67.5970	71.4838	75.2404	78.8895	82.4482	85.9295
31	69.0618	72.9841	76.7736	80.4536	84.0413	87.5501
32	70.5193	74.4766	78.2986	82.0090	85.6253	89.1611
33	71.9699	75.9616	79.8157	83.5560	87.2005	90.7631
34	73.4138	77.4396	81.3252	85.0950	88.7674	92.3564
35	74.8513	78.9107	82.8274	86.6265	90.3264	93.9413
36	76.2828	80.3753	84.3228	88.1506	91.8776	95.5183
37	77.7084	81.8336	85.8114	89.6677	93.4215	97.0876
38	79.1284	83.2859	87.2937	91.1781	94.9584	98.6495
39	80.5431	84.7324	88.7699	92.6820	96.4884	100.2043
40	81.9525	86.1733	90.2401	94.1796	98.0119	101.7524
41	83.3569	87.6089	91.7046	95.6713	99.5291	103.2938
42	84.7565	89.0393	93.1636	97.1571	101.0403	104.8289
43	86.1515	90.4647	94.6173	98.6374	102.5455	106.3579
44	87.5419	91.8853	96.0659	100.1122	104.0451	107.8809
45	88.9281	93.3012	97.5096	101.5818	105.5393	109.3983

m \ p	1	2	3	4	5	6
46	54.8778	63.2308	69.8331	75.5885	80.8239	85.7001
47	55.9721	64.4016	71.0595	76.8604	82.1351	87.0463
48	57.0653	65.5706	72.2835	78.1295	83.4430	88.3888
49	58.1576	66.7378	73.5052	79.3957	84.7478	89.7278
50	59.2490	67.9034	74.7246	80.6593	86.0495	91.0634
51	60.3394	69.0672	75.9419	81.9203	87.3482	92.3957
52	61.4290	70.2295	77.1570	83.1787	88.6440	93.7248
53	62.5177	71.3902	78.3701	84.4347	89.9370	95.0508
54	63.6055	72.5494	79.5812	85.6883	91.2273	96.3737
55	64.6926	73.7071	80.7904	86.9395	92.5149	97.6937
56	65.7788	74.8634	81.9976	88.1885	93.7999	99.0108
57	66.8643	76.0184	83.2030	89.4353	95.0824	100.3252
58	67.9490	77.1719	84.4067	90.6799	96.3625	101.6368
59	69.0330	78.3241	85.6085	91.9225	97.6402	102.9458
60	70.1163	79.4751	86.8087	93.1630	98.9155	104.2522
61	71.1989	80.6248	88.0072	94.4015	100.1886	105.5561
62	72.2808	81.7732	89.2041	95.6381	101.4595	106.8575
63	73.3620	82.9205	90.3994	96.8728	102.7282	108.1565
64	74.4426	84.0666	91.5932	98.1056	103.9948	109.4532

65	75.5226	85.2116	92.7855	99.3366	105.2593	110.7476
66	76.6020	86.3554	93.9763	100.5659	106.5219	112.0397
67	77.6807	87.4982	95.1656	101.7935	107.7824	113.3297
68	78.7589	88.6399	96.3536	103.0193	109.0410	114.6175
69	79.8365	89.7805	97.5402	104.2435	110.2978	115.9033
70	80.9135	90.9202	98.7254	105.4661	111.5527	117.1870
71	81.9900	92.0588	99.9093	106.6871	112.8058	118.4687
72	83.0659	93.1965	101.0919	107.9066	114.0571	119.7484
73	84.1413	94.3332	102.2733	109.1246	115.3067	121.0262
74	85.2162	95.4690	103.4534	110.3410	116.5546	122.3022
75	86.2906	96.6038	104.6323	111.5560	117.8009	123.5763
76	87.3645	97.7378	105.8100	112.7696	119.0455	124.8486
77	88.4379	98.8709	106.9866	113.9818	120.2885	126.1191
78	89.5108	100.0031	108.1620	115.1927	121.5300	127.3879
79	90.5833	101.1345	109.3363	116.4022	122.7699	128.6549
80	91.6553	102.2651	110.5095	117.6103	124.0084	129.9203
81	92.7268	103.3948	111.6816	118.8172	125.2453	131.1841
82	93.7979	104.5237	112.8527	120.0228	126.4808	132.4463
83	94.8686	105.6519	114.0227	121.2272	127.7149	133.7068
84	95.9389	106.7793	115.1917	122.4303	128.9475	134.9658
85	97.0087	107.9059	116.3597	123.6323	130.1788	136.2233
86	98.0781	109.0317	117.5267	124.8330	131.4088	137.4793
87	99.1472	110.1569	118.6927	126.0326	132.6374	138.7338
88	100.2158	111.2813	119.8578	127.2311	133.8647	139.9869
89	101.2840	112.4050	121.0220	128.4284	135.0908	141.2386
90	102.3519	113.5281	122.1852	129.6246	136.3156	142.4888

m \ p	7	8	9	10	11	12
46	90.3100	94.7126	98.9484	103.0464	107.0281	110.9101
47	91.6878	96.1197	100.3826	104.5060	108.5117	112.4165
48	93.0617	97.5225	101.8123	105.9609	109.9904	113.9178
49	94.4318	98.9212	103.2377	107.4113	111.4643	115.4140
50	95.7982	100.3160	104.6588	108.8571	112.9335	116.9053
51	97.1610	101.7068	106.0758	110.2986	114.3981	118.3919
52	98.5204	103.0940	107.4888	111.7359	115.8583	119.8738
53	99.8763	104.4775	108.8980	113.1691	117.3143	121.3513
54	101.2289	105.8574	110.3034	114.5984	118.7660	122.8244
55	102.5783	107.2339	111.7051	116.0238	120.2137	124.2933
56	103.9246	108.6071	113.1033	117.4454	121.6575	125.7580
57	105.2679	109.9770	114.4979	118.8633	123.0973	127.2187
58	106.6082	111.3437	115.8892	120.2777	124.5335	128.6756
59	107.9456	112.7073	117.2772	121.6865	125.9660	130.1285
60	109.2802	114.0679	118.6620	123.0960	127.3949	131.5778
61	110.6120	115.4256	120.0437	124.5002	128.8203	133.0234
62	111.9412	116.7803	121.4223	125.9011	130.2424	134.4655
63	113.2678	118.1323	122.7979	127.2989	131.6511	135.9041
64	114.5918	119.4815	124.1706	128.6936	133.0766	137.3394
65	115.9134	120.8281	125.5404	130.0853	134.4889	138.7713
66	117.2325	122.1720	126.9074	131.4741	135.8982	140.2000
67	118.5492	123.5133	128.2717	132.8599	137.3044	141.6255
68	119.8636	124.8522	129.6334	134.2430	138.7076	143.0480
69	121.1757	126.1886	130.9924	135.6233	140.1080	144.4674
70	122.4856	127.5226	132.3489	137.0009	141.5055	145.8838
71	123.7933	128.8543	133.7029	138.3758	142.9003	147.2974
72	125.0988	130.1837	135.0545	139.7482	144.2923	148.7081
73	126.4023	131.5108	136.4036	141.1180	145.6817	150.1161
74	127.7037	132.8357	137.7505	142.4854	147.0685	151.5213
75	129.0031	134.1585	139.0950	143.8503	148.4527	152.9238
76	130.3006	135.4791	140.4372	145.2129	149.8344	154.3238
77	131.5961	136.7977	141.7772	146.5731	151.2137	155.7211
78	132.8897	138.1142	143.1151	147.9310	152.5906	157.1160
79	134.1815	139.4287	144.4509	149.2867	153.9651	158.5084
80	135.4714	140.7413	145.7845	150.6401	155.3372	159.8983

81	136.7596	142.0520	147.1161	151.9914	156.7071	161.2859
82	138.0460	143.3607	148.4457	153.3406	158.0748	162.6711
83	139.3306	144.6676	149.7733	154.6876	159.4403	164.0540
84	140.6136	145.9727	151.0990	156.0327	160.8036	165.4347
85	141.8949	147.2760	152.4228	157.3757	162.1648	166.8132
86	143.1746	148.5775	153.7447	158.7167	163.5239	168.1895
87	144.4527	149.8773	155.0647	160.0558	164.8809	169.5636
88	145.7292	151.1754	156.3830	161.3930	166.2360	170.9357
89	147.0041	152.4719	157.6995	162.7282	167.5891	172.3056
90	148.2775	153.7667	159.0142	164.0617	168.9402	173.6736

p / m	1	2	3	4	5	6
91	103.4193	114.6504	123.3476	130.8198	137.5391	143.7377
92	104.4864	115.7721	124.5090	132.0139	138.7614	144.9852
93	105.5532	116.8931	125.6696	133.2069	139.9826	146.2314
94	106.6195	118.0135	126.8293	134.3989	141.2025	147.4762
95	107.6855	119.1332	127.9882	135.5899	142.4213	148.7198
96	108.7512	120.2523	129.1462	136.7799	143.6389	149.9621
97	109.8165	121.3708	130.3034	137.9689	144.8554	151.2032
98	110.8815	122.4886	131.4598	139.1569	146.0708	152.4430
99	111.9462	123.6059	132.6154	140.3440	147.2851	153.6816
100	113.0105	124.7226	133.7703	141.5301	148.4984	154.9190
101	114.0745	125.8387	134.9243	142.7153	149.7106	156.1553
102	115.1382	126.9542	136.0776	143.8996	150.9217	157.3904
103	116.2016	128.0691	137.2302	145.0830	152.1318	158.6243
104	117.2647	129.1835	138.3820	146.2655	153.3409	159.8571
105	118.3275	130.2974	139.5331	147.4472	154.5490	161.0889
106	119.3899	131.4107	140.6834	148.6280	155.7561	162.3195
107	120.4521	132.5234	141.8331	149.8079	156.9623	163.5490
108	121.5140	133.6357	142.9820	150.9870	158.1674	164.7775
109	122.5756	134.7474	144.1303	152.1653	159.3717	166.0049
110	123.6369	135.8586	145.2779	153.3427	160.5750	167.2313
111	124.6980	136.9693	146.4248	154.5194	161.7774	168.4566
112	125.7588	138.0796	147.5711	155.6952	162.9789	169.6810
113	126.8192	139.1893	148.7167	156.8703	164.1795	170.9044
114	127.8795	140.2985	149.8617	158.0446	165.3792	172.1268
115	128.9394	141.4072	151.0060	159.2182	166.5780	173.3482
116	129.9991	142.5155	152.1497	160.3910	167.7760	174.5686
117	131.0585	143.6233	153.2928	161.5630	168.9731	175.7882
118	132.1177	144.7307	154.4353	162.7344	170.1694	177.0067
119	133.1767	145.8376	155.5772	163.9050	171.3649	178.2244
120	134.2354	146.9440	156.7185	165.0749	172.5595	179.4412
121	135.2939	148.0500	157.8592	166.2440	173.7534	180.6570
122	136.3521	149.1556	158.9993	167.4125	174.9464	181.8720
123	137.4100	150.2608	160.1388	168.5803	176.1387	183.0861
124	138.4677	151.3655	161.2778	169.7474	177.3302	184.2994
125	139.5252	152.4698	162.4162	170.9139	178.5209	185.5117
126	140.5825	153.5736	163.5541	172.0797	179.7109	186.7233
127	141.6395	154.6771	164.6914	173.2448	180.9001	187.9340
128	142.6963	155.7802	165.8282	174.4093	182.0885	189.1439
129	143.7529	156.8828	166.9645	175.5731	183.2763	190.3530
130	144.8093	157.9851	168.1002	176.7363	184.4633	191.5612
131	145.8654	159.0870	169.2354	177.8989	185.6496	192.7687
132	146.9214	160.1884	170.3701	179.0609	186.8352	193.9754
133	147.9771	161.2895	171.5043	180.2222	188.0201	195.1813
134	149.0326	162.3902	172.6379	181.3830	189.2043	196.3865
135	150.0879	163.4906	173.7711	182.5431	190.3878	197.5908

p / m	7	8	9	10	11	12
91	149.5494	155.0598	160.3272	165.3933	170.2894	175.0395
92	150.8199	156.3514	161.6385	166.7231	171.6367	176.4035

93	152.0889	157.6415	162.9481	168.0512	172.9822	177.7655
94	153.3564	158.9299	164.2561	169.3775	174.3259	179.1256
95	154.6226	160.2169	165.5625	170.7021	175.6677	180.4839
96	155.8873	161.5024	166.8673	172.0250	177.0078	181.8403
97	157.1507	162.7864	168.1705	173.3463	178.3462	183.1949
98	158.4128	164.0690	169.4722	174.6660	179.6828	184.5476
99	159.6735	165.3501	170.7724	175.9840	181.0178	185.8987
100	160.9330	166.6299	172.0710	177.3005	182.3511	187.2480
101	162.1911	167.9082	173.3682	178.6154	183.6827	188.5955
102	163.4480	169.1852	174.6640	179.9288	185.0128	189.9414
103	164.7036	170.4609	175.9583	181.2407	186.3412	191.2856
104	165.9580	171.7352	177.2512	182.5510	187.6681	192.6282
105	167.2112	173.0082	178.5428	183.8599	188.9934	193.9692
106	168.4632	174.2800	179.8329	185.1674	190.3173	195.3086
107	169.7141	175.5505	181.1217	186.4734	191.6396	196.6464
108	170.9637	176.8197	182.4092	187.7781	192.9604	197.9826
109	172.2123	178.0877	183.6954	189.0813	194.2798	199.3173
110	173.4596	179.3545	184.9802	190.3831	195.5977	200.6505
111	174.7059	180.6201	186.2638	191.6837	196.9142	201.9823
112	175.9511	181.8845	187.5461	192.9828	198.2294	203.3125
113	177.1952	183.1477	188.8272	194.2807	199.5431	204.6413
114	178.4382	184.4098	190.1071	195.5772	200.8554	205.9687
115	179.6802	185.6707	191.3857	196.8725	202.1665	207.2947
116	180.9211	186.9306	192.6632	198.1665	203.4761	208.6193
117	182.1610	188.1893	193.9394	199.4593	204.7845	209.9424
118	183.3999	189.4469	195.2145	200.7508	206.0916	211.2643
119	184.6378	190.7035	196.4885	202.0411	207.3974	212.5848
120	185.8747	191.9589	197.7613	203.3303	208.7019	213.9040
121	187.1106	193.2134	199.0330	204.6182	210.0051	215.2218
122	188.3456	194.4667	200.3036	205.9049	211.3072	216.5384
123	189.5796	195.7191	201.5731	207.1905	212.6080	217.8537
124	190.8126	196.9704	202.8415	208.4749	213.9076	219.1677
125	192.0447	198.2208	204.1088	209.7582	215.2060	220.4805
126	193.2759	199.4701	205.3751	211.0404	216.5033	221.7921
127	194.5062	200.7185	206.6403	212.3215	217.7993	223.1024
128	195.7356	201.9659	207.9045	213.6015	219.0943	224.4116
129	196.9641	203.2123	209.1677	214.8804	220.3881	225.7195
130	198.1918	204.4578	210.4298	216.1583	221.6807	227.0263
131	199.4185	205.7024	211.6910	217.4350	222.9723	228.3319
132	200.6444	206.9460	212.9512	218.7108	224.2628	229.6364
133	201.8695	208.1888	214.2104	219.9855	225.5522	230.9397
134	203.0937	209.4306	215.4687	221.2592	226.8405	232.2419
135	204.3171	210.6715	216.7260	222.5319	228.1277	233.5430

p / m	1	2	3	4	5	6
136	151.1431	164.5906	174.9038	183.7027	191.5706	198.7945
137	152.1980	165.6901	176.0360	184.8617	192.7528	199.9974
138	153.2527	166.7894	177.1677	186.0201	193.9343	201.1995
139	154.3072	167.8883	178.2990	187.1779	195.1152	202.4009
140	155.3615	168.9868	179.4297	188.3352	196.2954	203.6016
141	156.4156	170.0849	180.5600	189.4919	197.4750	204.8016
142	157.4696	171.1828	181.6899	190.6480	198.6539	206.0009
143	158.5233	172.2802	182.8193	191.8036	199.8322	207.1995
144	159.5769	173.3774	183.9482	192.9587	201.0099	208.3974
145	160.6302	174.4741	185.0767	194.1132	202.1870	209.5946
146	161.6834	175.5706	186.2047	195.2672	203.3635	210.7912
147	162.7364	176.6668	187.3324	196.4207	204.5393	211.9870
148	163.7892	177.7626	188.4595	197.5736	205.7146	213.1822
149	164.8418	178.8580	189.5863	198.7261	206.8893	214.3768
150	165.8943	179.9532	190.7126	199.8780	208.0634	215.5707

m \ p	7	8	9	10	11	12
136	205.5396	211.9116	217.9823	223.8036	229.4139	234.8430
137	206.7614	213.1507	219.2377	225.0743	230.6991	236.1419
138	207.9823	214.3390	220.4922	226.3440	231.9832	237.4397
139	209.2025	215.6265	221.7458	227.6128	233.2663	238.7365
140	210.4218	216.8631	222.9985	228.8806	234.5484	240.0322
141	211.6404	218.0988	224.2503	230.1474	235.8295	241.3269
142	212.8582	219.3338	225.5012	231.4133	237.1097	242.6206
143	214.0753	220.5679	226.7512	232.6783	238.3888	243.9132
144	215.2916	221.8012	228.0004	233.9424	239.6670	245.2048
145	216.5072	223.0337	229.2487	235.2056	240.9443	246.4954
146	217.7220	224.2654	230.4962	236.4679	242.2206	247.7851
147	218.9361	225.4963	231.7428	237.7293	243.4960	249.0737
148	220.1494	226.7265	232.9886	238.9898	244.7704	250.3614
149	221.3621	227.9558	234.2336	240.2494	246.0439	251.6482
150	222.5740	229.1845	235.4777	241.5082	247.3165	252.9339

Weighted PCM*

EDWARD BEDROSIAN†

Summary—**A modified form of pulse-code modulation, called weighted pcm, is described in which the relative amplitudes of the pulses within the pulse-code groups are adjusted so as to minimize the noise power in the reconstructed signal due to errors in transmission. A performance analysis shows the knee of the output signal-to-noise ratio curve to be moved 1.4 db to the left for a weighted seven-digit pcm system. An information rate study reveals that the maximum improvement which can ever be achieved by any encoding process over a conventional seven-digit pcm system is only 8 db. The importance of selecting a suitable system worth criterion is emphasized by showing that weighting increases the information rate relative to an rms fidelity criterion but decreases it on a pure equivocation basis.**

Introduction

SINCE its invention in 1939 [1] pulse-code modulation, or pcm, has received considerable attention because of its remarkable noise-cleaning property. The principles and practices of pcm are well treated in the literature [2]–[5] and will not be reviewed here. It is merely necessary to note that the message signal is sampled periodically and that the samples are quantized to a set of discrete levels. The quantized samples are expressed as binary numbers and represented electrically as pulse sequences which are transmitted on the (noisy) channel to a receiver which reverses the procedure.

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A conventional pcm transmission consists of equal-amplitude pulses; thus, the probability of error is the same for all the pulse positions. However, within a given pulse-code group it is obvious that the various pulses are of different importance to the reconstructed signal value since each pulse denotes the presence or absence of a different power of the binary base 2. It would appear that an improved performance could be obtained by "weighting" the various pulses, *i.e.*, adjusting their relative amplitudes, in such a way as to let the probability of an error be in keeping with the value of the pulse.

The Error Noise Power

Let the pulse positions in a typical pulse-code group be denoted by the integers 1 to n and let the corresponding binary numbers 0 and 1 be denoted by positive and negative pulses, the amplitude of the ith pulse being a_i. If the probability density of the signal amplitudes is uniform (or made so by compression), then the signal will occupy all quantization levels with equal likelihood and it can be reasoned that the mean value of the ensemble of pulse sequences will be zero.

A simplified noise model is assumed wherein a random noise having a normal distribution of amplitudes is added linearly to the pulses. Also, the noise amplitudes at the various pulse positions are assumed to be uncorrelated. (This model corresponds quite closely to a receiver using synchronous detection.) The decision process can then

be evaluated by noting that the probability p_i of making an error at the ith pulse position can be written

$$p_i = \frac{1}{\sqrt{2\pi}} \int_{a_i}^{\infty} e^{-x^2/2}\, dx, \tag{1}$$

where the standard deviation of the noise, and hence the noise power, has been taken as unity. The average signal power becomes

$$S = \frac{1}{n} \sum_{i=1}^{n} a_i^2, \tag{2}$$

since it is merely the mean-square pulse amplitude.

Errors will be rare in an operating system so it will be assumed that $p_i \ll 1$ and that the occurrence of more than one error in a given pulse-code group can be disregarded. The probability that there will be just one error and that it will occur at the ith pulse position is given by

$$p_i(1) = p_i \prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - p_j) \approx p_i.$$

Let E_i denote the error which occurs in the reconstructed signal when there is an error at the ith pulse position. Since errors are equally likely to result from noise adding to or subtracting from the pulses and since the pulses denote binary digits, it follows that

$$E_i = \pm 2^{i-1}, \qquad E_i^2 = 4^{i-1}.$$

The noise power N_e in the output due to an error is the mean value of E_i^2 over the pulse group, or

$$N_e = \sum_{i=1}^{n} p_i(1) E_i^2 = \sum_{i=1}^{n} p_i 4^{i-1}. \tag{3}$$

The Minimization Problem

The weighting function for the a_i is found by minimizing the error noise power of (3) subject to the constraint of constant power given by (2). Consider the function

$$F(x_1, x_2, \cdots, x_n, \lambda) = f(x_1, x_2, \cdots, x_n) + \lambda g(x_1, x_2, \cdots, x_n)$$

where λ is the Lagrange multiplier and where the variables of the function f are connected by the condition $g = 0$. By the method of Lagrange's undetermined multipliers [6], the set of $n + 1$ equations in the x_i and λ given by

$$\frac{\partial F}{\partial x_i} = 0, \qquad g = 0$$

is satisfied at the extreme values of f provided that not all the derivatives $\partial g/\partial x_i$ vanish. From (2) and (3) it is seen that

$$f(a_1, a_2, \cdots, a_n) = \sum_{i=1}^{n} p_i 4^{i-1}$$

$$g(a_1, a_2, \cdots, a_n) = S - \frac{1}{n} \sum_{i=1}^{n} a_i^2 = 0$$

yielding

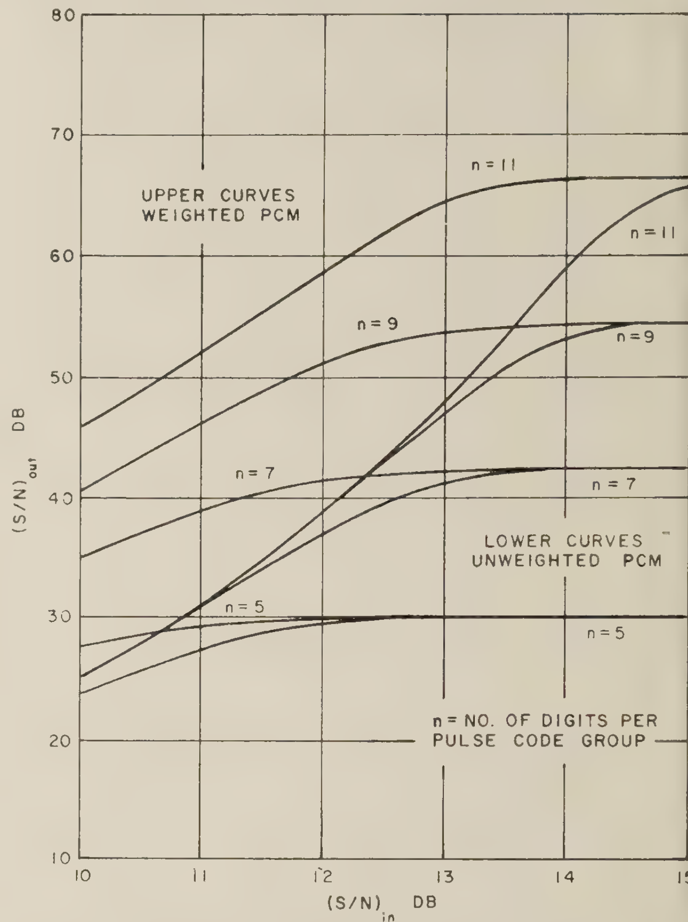

Fig. 1—Comparison of weighted and unweighted pcm systems.

$$4^{i-1} \frac{dp_i}{da_i} - \frac{2}{n} a_i \lambda = 0, \qquad S - \frac{1}{n} \sum_{i=1}^{n} a_i^2 = 0. \tag{4}$$

It can be shown that a real, positive, unique solution to (4) exists and yields a true minimum. The resulting weighting function is well approximated by

$$a_i^2 = S + \frac{S}{1 + S}\left(i - \frac{n+1}{2}\right) \ln 16 \tag{5}$$

which is explicit and seen to depend only on the average signal power and the number of digits per pulse-code group.

System Performance

The compressed message signal was assumed to be distributed uniformly over the quantization levels. If the quantization steps are given unity amplitude the output signal power becomes

$$S_0 = \frac{4^n - 1}{12} \tag{6}$$

which is the resulting mean-square value.

The quantization noise power is found by assuming that the quantization error amplitudes are uniformly distributed in all the quantization levels yielding

$$N_q = \int_{-1/2}^{+1/2} x^2\, dx = \frac{1}{12}. \tag{7}$$

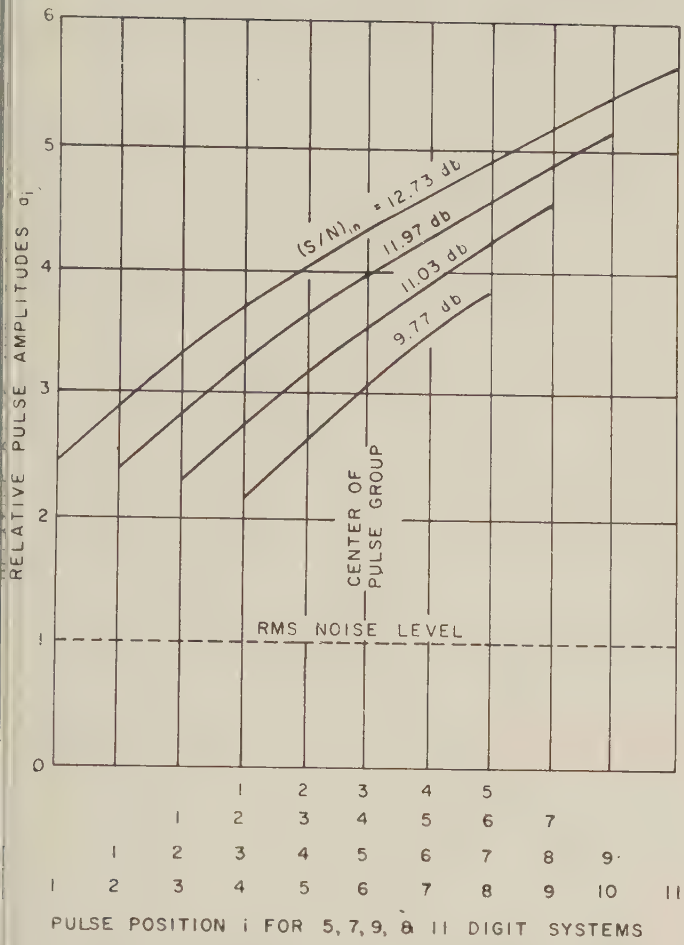

Fig. 2—Weighting function for optimum performance at design input signal-to-noise ratio.

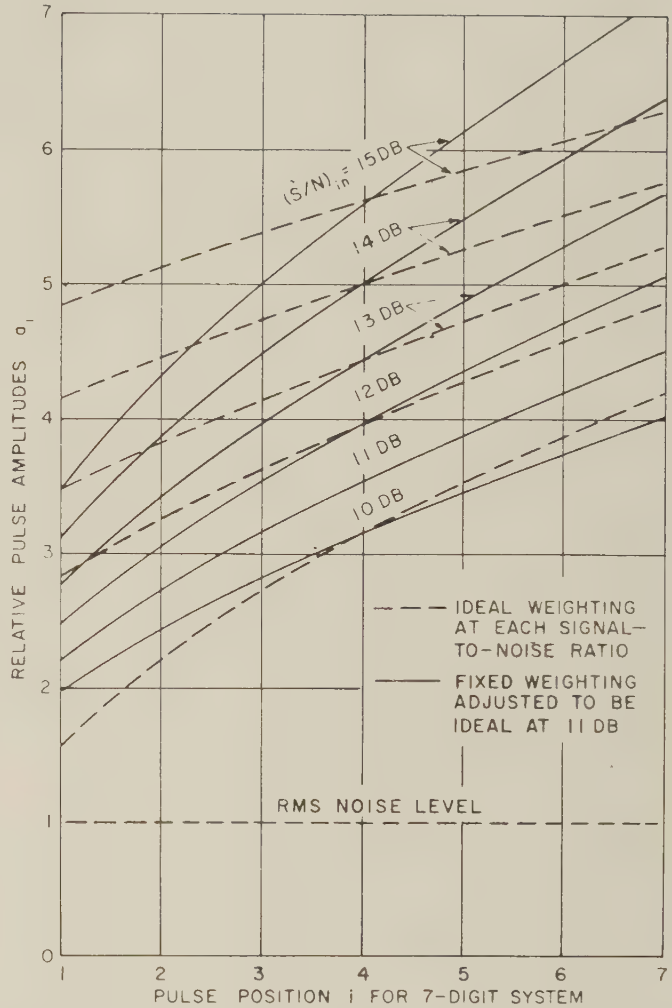

Fig. 3—Comparison of relative pulse amplitudes for practical and optimum seven-digit pcm systems.

The output signal-to-noise ratio then is written

$$\left(\frac{S}{N}\right)_{\text{out}} = \frac{S_0}{N_e + N_q}, \tag{8}$$

where the noise powers are assumed to be uncorrelated so they can be added directly.

The output of a conventional pcm system differs only with respect to the error noise power. The expression given by (3) is still valid but with the simplification that the p_i are equal since the a_i are equal. The error noise power then can be written

$$N_e' = p \sum_{i=1}^{n} 4^{i-1} = \frac{4^n - 1}{3} p, \tag{9}$$

where the prime denotes the unweighted system. The output signal-to-noise ratio becomes

$$\left(\frac{S}{N}\right)_{\text{out}}' = \frac{S_0}{N_e' + N_q}. \tag{10}$$

The performance of conventional and weighted pcm systems are compared in Fig. 1 for various numbers of digits per pulse-code group.

The Practical Weighted System

A practical system cannot achieve the weighting given by (5) continuously since the pulse amplitude distribution then would become a function of the input signal power to the receiver, a quantity which fluctuates greatly in normal service. A practical solution is to select for the a_i those values for a particular value of S and to accept the nonoptimum performance which occurs at other values of S.

A logical choice for this design point is the value of input signal power at which the quantization and error noise powers are equal. This point is significant since for larger input signal powers the output noise power is determined principally by the quantization noise, while at smaller input signal powers the output noise power rises so rapidly that the system is virtually useless anyway. The relative pulse amplitudes corresponding to these design values have been computed using (5) and are shown in Fig. 2. These are actual weighting functions to be applied to the pulses at the transmitter.

The effect of fixed weighting on the relative pulse amplitudes for a seven-digit system is illustrated in Fig. 3. The dotted lines show the weighting to be used at each input signal-to-noise ratio to achieve optimum performance. The solid lines denote the fixed weighting which is adjusted to be optimum at the design input signal-to-noise ratio of approximately 11 db. It is seen that the optimum

weighting tends to level off at high input signal-to-noise ratios while the fixed weighting maintains the amplitude ratios established at the design input signal-to-noise ratio. The effect on the output signal-to-noise ratio is only apparent, however, since quantization noise soon predominates in either case.

The performance of the practical system is computed in a fashion similar to the preceding computations in that the output signal-to-noise ratio is written again as

$$\left(\frac{S}{N}\right)''_{\text{out}} = \frac{S_0}{N''_e + N_q}, \tag{11}$$

except that the double prime on the error noise power denotes the use of the fixed pulse amplitude distribution. This noise power is given by the general form, (3), in which the p_i are determined from the actual a_i for each value of S as in Fig. 3 rather than from the optimum condition given by (5). The performance of a practical seven-digit weighted pcm system is shown in Fig. 4. It is seen that the effect of fixing the relative pulse amplitudes to those determined at the design point is quite small compared with the general improvement of performance achieved by weighting.

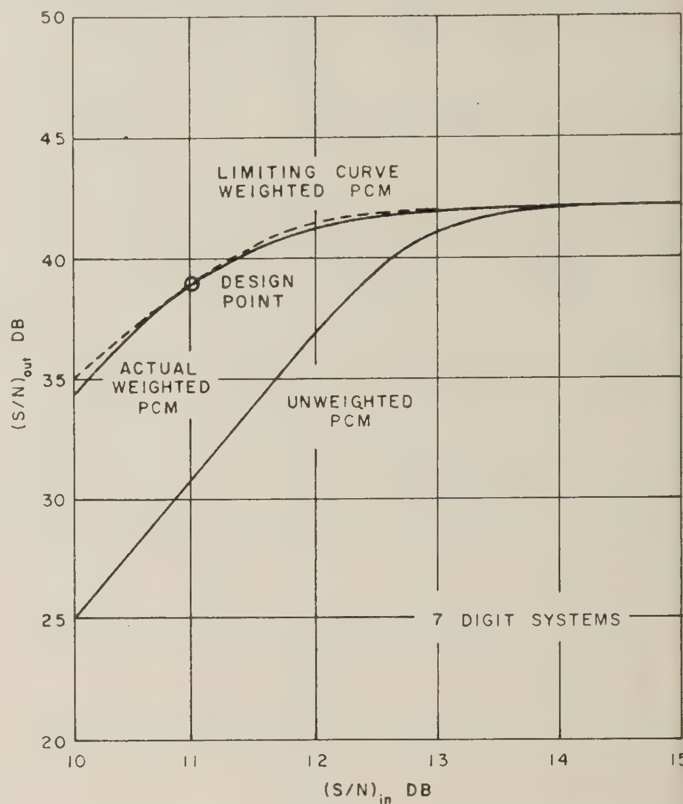

Fig. 4—Performance characteristics of a typical weighted pcm system.

Information Rate

According to Shannon (Theorem 23), [7], information rate for any continuous source of average power Q and band W relative to rms measure of fidelity is bounded by

$$W \log \frac{Q_1}{N} \leq R \leq W \log \frac{Q}{N}, \tag{12}$$

where Q_1 is the entropy power of the source and N is the allowed mean-square error between the original and recovered messages.

The entropy power of a source is defined as the average power of a normally distributed source having the same entropy as the original source. The pcm systems discussed here are assumed to have uniform distributions of signal amplitudes. Let x denote a typical signal amplitude which lies in the interval $(-a, a)$ with a probability density

$$p(x) = \begin{cases} \dfrac{1}{2a}, & |x| \leq a \\ 0, & |x| > a. \end{cases}$$

The source power Q is the mean-square value of x or

$$Q = \int_{-\infty}^{\infty} x^2 p(x)\, dx = \frac{a^2}{3}.$$

The entropy of the source is

$$H = -\int_{-\infty}^{\infty} p(x) \log p(x)\, dx = \log 2\sqrt{3Q}. \tag{13}$$

A normally distributed source has a probability density

$$p(y) = \frac{1}{\sqrt{2\pi Q_1}} e^{-y^2/2Q_1},$$

where Q_1 is its average power. Its entropy is

$$H_1 = -\int_{-\infty}^{\infty} p(y) \log p(y)\, dy = \log \sqrt{2\pi e Q_1}. \tag{14}$$

Equating the entropies of (13) and (14) gives

$$Q_1 = \frac{6}{\pi e} Q, \tag{15}$$

as the entropy power of the uniformly distributed pcm source.

The allowed mean-square error N in (12) is simply the sum of the quantization and error noise powers while the source band refers to the signal prior to coding as a pcm signal. With the help of (15), the bounded information rate of (12) can be written

$$\frac{1}{2n} \log \frac{6}{\pi e}\left(\frac{S}{N}\right)_{\text{out}} \leq \frac{R}{2nW} \leq \frac{1}{2n} \log \left(\frac{S}{N}\right)_{\text{out}}, \tag{16}$$

where $R/2nW$ has the units of bits per symbol when the logarithm is taken to base 2 since a pcm channel transmits $2nW$ symbols (or pulses) per second.

By Shannon's Theorem 17, the capacity of a channel of band W perturbed by white thermal noise of power N when the average transmitter power is limited to P is given by

$$C = W \log\left(1 + \frac{P}{N}\right),$$

where the capacity is the maximum information rate possible. For a channel of width nW cps the capacity becomes

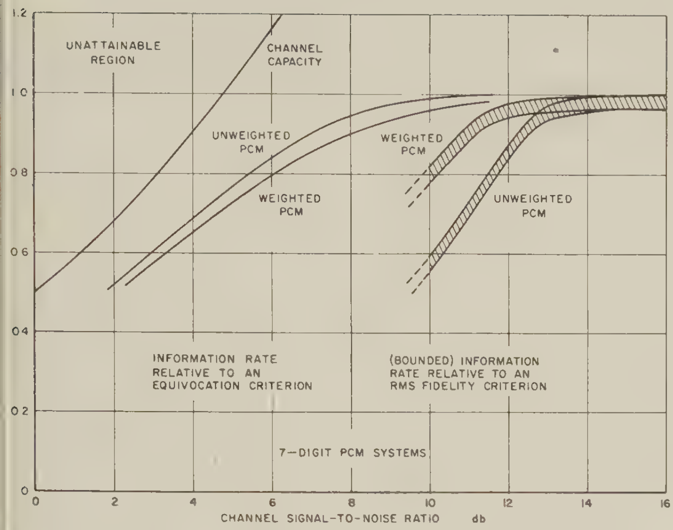

Fig. 5—Information rates for typical weighted and unweighted pcm systems.

$$\frac{C}{2nW} = \frac{1}{2} \log \left[1 + \left(\frac{S}{N} \right)_{\text{in}} \right] \tag{17}$$

bits per symbol.

The channel capacity of (17) and the bounded information rate of (16) are plotted in Fig. 5 for the seven-digit system of Fig. 4. This plot not only demonstrates the essential shape of the curves of Fig. 4 by indicating an input signal-to-noise ratio below which the performance deteriorates rapidly, but it also shows how good pcm is when compared with the performance of the "ultimate" system. Also, it shows that the process of weighting achieves a significant improvement in terms of the maximum improvement possible. According to Fig. 4, weighting moves the knee of the output signal-to-noise ratio curve about 1.4 db to the left, an unimpressive amount in itself. On the other hand, Fig. 5 shows that the knee of the information-rate curve has moved 1.4 db out of a maximum possible 8 db. Generally it is conceded that significant approaches to the channel capacity cannot be made except by recourse to considerable circuitry complication and message delay.

It is instructive to compute the information rate on the basis of the per-symbol equivocation considering the transmission channel alone. The source is a device generating positive and negative pulses with equal likelihood giving a source entropy of one bit per symbol. In a conventional pcm transmission all pulses have the same probability p of being in error so the equivocation is simply[7]

$$H_y(x) = -[p \log p + (1 - p) \log (1 - p)]. \tag{18}$$

For weighted pcm the pulses within the pulse groups have differing error probabilities so the equivocation becomes

$$H_y(x) = -\frac{1}{n} \sum_{i=1}^{n} [p_i \log p_i + (1 - p_i) \log (1 - p_i)]. \tag{19}$$

The information rates $[H(x) - H_y(x)]$ for the seven-digit systems considered are also shown in Fig. 5. First, it is clear that there is no quantitative relationship between the rms fidelity and equivocation criterion rates in the region where errors are significant. Second, it is seen that weighting increases the information rate relative to an rms fidelity criterion but decreases it on an equivocation basis. This fact is not startling when it is recalled that the purpose of weighting is to decrease the net effect of errors rather than their total number, but it is disturbing when one realizes that exactly opposite effects can be predicted if care is not exercised in selecting the criterion on which a system is to be judged.

Conclusion

A well-designed pcm communications system invariably includes a comfortable safety margin, *i.e.*, an average input signal-to-noise ratio 20 to 30 db above the threshold, to protect the link from failure in all but the deepest fades. Under such conditions the 1.4 db or so by which weighting reduces the threshold may not be an important enough factor to justify its use if excessive circuitry complications result. It may be that a scheme in which, for example, the weighted pulses are transmitted in frequency-division-multiplex does not constitute such a complication, but the acceptance of a technique such as weighting must depend ultimately on the more experienced considerations of the practicing systems engineers. Regardless of its practicability, however, the process itself is of interest theoretically because it improves the system performance without recourse to redundancy and processing-time lags for error correcting.

The information rate study shows that the 1.4-db improvement mentioned above represents a significant approach to the channel capacity. Furthermore, it brings out the importance of looking beyond the transmission channel itself when selecting the system worth criterion.

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[2] Marton, L. *Advances in Electronics*, New York: Academic Press, Inc., Vol. 3; 1951. See H. F. Mayer, "Principles of Pulse Code Modulation."

[3] Black, H. S. *Modulation Theory*, New York: D. Van Nostrand Co., Inc., ch. 19; 1953.

[4] Bennett, W. R. "Spectra of Quantized Signals," *Bell System Technical Journal*, Vol. 27 (July, 1948), pp. 446–472.

[5] Oliver, B. M., Pierce, J. R., and Shannon, C. E. "The Philosophy of PCM," Proceedings of the IRE, Vol. 36 (November, 1948), pp. 1324–1331.

[6] Courant, R. *Differential and Integral Calculus*, New York: Nordeman Publishing Co., Inc., Vol. 2, p. 198; 1937. The rule merely gives a necessary condition for the occurrence of an extreme value.

[7] Shannon, C. E. "A Mathematical Theory of Communication," *Bell System Technical Journal*, Vol. 27 (July, 1948), pp. 379–423, and Vol. 27 (October, 1948), pp. 623–656.

[8] Chang, S. S. L., *et al.* "Supplementary Notes on Evaluation and Optimization of Digital Communication Systems," New York University, New York, N. Y., College of Eng., First Sci. Rep.; January 15, 1957 to April 15, 1957. AFCRC-TN-57-355, AD 117047. The method of linear programming is applied to messages containing symbols of unequal importance to maximize information rate subject to constraints on power, bandwidth, time, and reliability. The concept of "the utility of a digit in relation to other digits of the same number" is introduced and used to formulate a functional which when maximized will maximize the total utility value of the message.

Radar Detection Probability with Logarithmic Detectors*

BEN A. GREEN, JR.†

Summary—It is shown that use of a logarithmic (rather than square-law) detector in a search radar system inflicts a loss of sensitivity equivalent to a power loss of the order of one db under typical conditions. Curves of probability of detection vs relative range are given for various false alarm probabilities and various numbers of pulses integrated. The power loss (in db) is roughly proportional to the logarithm of the number of pulses integrated.

Introduction

A LOGARITHMIC detector is a device which abstracts from an incoming high-frequency signal the logarithm of its envelope. Radar receivers employing a logarithmic detector are interesting to radar designers because they provide wide "dynamic range" and because they facilitate analog calculation of products, ratios, and powers.

But one may wonder what effect this type of detector has on the "sensitivity" of the receiver; more precisely, what happens to the probability of detection for a fixed probability of false alarm? If only one pulse is considered at a time, then the answer is that the sensitivity is precisely the same as that when any ordinary detector is used. But a difference appears when pulses are "integrated," or summed, after detection in order to enhance the snr. The purpose of this work was to calculate the difference.

This work is an extension of the work of Marcum,[1] who calculated the probability of detection after integration of pulses from a "square-law" detector.

A paper by Croney[2] gives the mean and standard deviation of the output of a logarithmic receiver when the input is thermal noise. The present work extends the calculation to higher moments of the distribution.

Our approach is to approximate the distribution of the integrated pulses *when they contain a signal* by a normal distribution, whose parameters are calculated; *when they do not contain a signal* the distribution is calculated numerically or approximated by an Edgeworth series, depending on the number of pulses integrated.

Calculations

We write the probability density for the amplitude of the envelope of a high-frequency signal containing a sine wave of amplitude a and thermal noise of unit rms amplitude as follows:[3]

$$P(x|a) = xI_0(ax)\exp\left(-\frac{x^2+a^2}{2}\right),$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero. (We allow only positive x.) From this we may, when necessary, calculate $Q(x|a)$, the corresponding probability density for the logarithm of this envelope. Further we may calculate $Q_k(x|a)$, that for the sum of k such pulses are samples of the envelope.

The usual detection philosophy is to assign a certain small probability P_N to the occurrence of a *false* alarm during one independent time interval. Then one agrees to announce the presence of a signal whenever the receiver output exceeds a certain threshold V, set to make the probability that noise alone will exceed it equal to P_N. Thus, V is set from

$$P_N = \int_V^\infty Q_k(x|0)\,dx.$$

Then the probability P_D that a sine wave of amplitude a will cause the alarm to be sounded is

$$P_D = \int_V^\infty Q_k(x|a)\,dx.$$

We now consider the first part of the problem: to calculate the threshold V. For pure noise, $a = 0$, and we have

$$P(x|0) = x\exp(-x^2/2).$$

If we call $y = \ln x$, then by a standard argument,

$$Q(y|0)\,dy = P(x|0)\,dx.$$

Thus,

$$Q(y|0) = \exp(2y - 1/2e^{2y})$$

where y can be positive or negative.

To calculate $Q_{10}(x|0)$, the density for the sum of ten noise pulses, we resorted to numerical integration, a ten-fold convolution[4] of $Q(x|0)$ with itself. This was carried out on the Bendix G-15 general purpose digital computer by the Bendix Radio computing staff.

To calculate $Q_{100}(x|0)$ numerically was not feasible. It was possible, however, to employ the Edgeworth series,[5] an expansion of the density in terms of the normal distribution density and its derivatives. It was proved that the "cumulants"[6] of $Q(x|0)$ were given by[7]

$$x_n = (-2)^{-n}\cdot\zeta(n)\cdot(n-1)!$$

* Manuscript received by the PGIT, September 12, 1957. This work was performed by Bendix Radio Corp. under contract with Cornell Aeronautical Labs.

† Electro Metallurgical Co., Niagara Falls, N. Y. Formerly with Bendix Radio Corp., Towson, Md.

[1] J. I. Marcum, "A Statistical Theory of Target Detection by Pulsed Radar," U.S.A.F., Project RAND, RAND Corp., Santa Monica, Calif., Res. Memo. RM-754; 1947.

[2] J. Croney, "Clutter on radar displays," *Wireless Eng.*, vol. 33, pp. 83–96; April, 1956.

[3] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 24, p. 101; January, 1945.

[4] H. Cramer, "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J., p. 191; 1951.

[5] *Ibid*, p. 229.

[6] *Ibid*, p. 186.

[7] Derivation due to James Dokas, Bendix Radio Corp., Towson, Md.

here $\zeta(n)$ is the Riemann zeta function. By such means $_{100}(x|0)$ was approximated sufficiently well to calculate he threshold for $P_N \doteq 10^{-5}$ and 10^{-7}. These results were xtrapolated for $P_N = 10^{-10}$, since the series begins to iverge, but the threshold is relatively insensitive to P_N.

Now for the second half of the problem: to calculate P_D, the probability of detection for various values of P_N nd the signal amplitude a. It may be shown that as a rows large, $P(x|0)$ approaches a normal distribution ensity with mean, a. Further, the density $Q(x|a)$ becomes ormal in the vicinity of $x = \ln a$. And, finally, since he sum of several identical variables is more nearly ormally distributed than any one, we feel safe in approxi- ating $Q_k(x|a)$ by a normal density. This approximation s best near $x = \ln a$; its failure becomes evident in the ails. In what follows we claim significant results only ver the range,

$$0.01 < P_D < 0.99.$$

his is not a serious limitation.

One must then calculate the mean m and standard eviation σ of $Q(x|a)$. It is simpler to calculate

$$M_1 = \int_0^\infty (\ln x)P(x|a)\,dx$$

$$M_2 = \int_0^\infty (\ln x)^2P(x|a)\,dx,$$

nd then assert

$$m = M_1, \quad \text{and} \quad \sigma^2 = M_2 - M_1^2.$$

Surprisingly enough, it is not difficult to get a series epresentation of M_1 and M_2 in powers of a^2. Briefly, ne uses Rice's expression[8] for the moments of $P(x|a)$

$$f(n) = \int_0^\infty x^nP(x|a)\,dx$$

$$= 2^{n/2}\cdot\Gamma(n/2+1)\cdot{}_1F_1(-n/2;1;-a^2/2).$$

his is an entire function of n;[9] thus, we may differentiate ith respect to n and set n equal to zero. This gives M_1, s may be seen by inspection. The second derivative valuated at $n = 0$ gives M_2. The series for the hyper- eometric function was differentiated term by term with espect to n, giving rise to a series for M_1 and M_2. In fact

$$M_1 = \ln a + 1/2\int_{a^2/2}^\infty \frac{e^{-t}}{t}\,dt.$$

he integral is tabulated.[10] M_2 could not be expressed in erms of tabulated functions. For large values of a, the ean approaches $\ln a$ and the standard deviation ap- roaches a^{-1}. Fig. 1 shows the mean and standard deviation nd their asymptotes.

[8] Rice, *op. cit.*, p. 101.

[9] H. Bateman, "Higher Transcendental Functions," McGraw- ill Book Co., Inc., New York, N. Y., vol. 1, p. 260; 1953. Compiled Bateman Manuscript Project Staff.

[10] E. Jahnke and F. Emde, "Tables of Functions," Dover Publi- tions, New York, N. Y., p. 6; 1945.

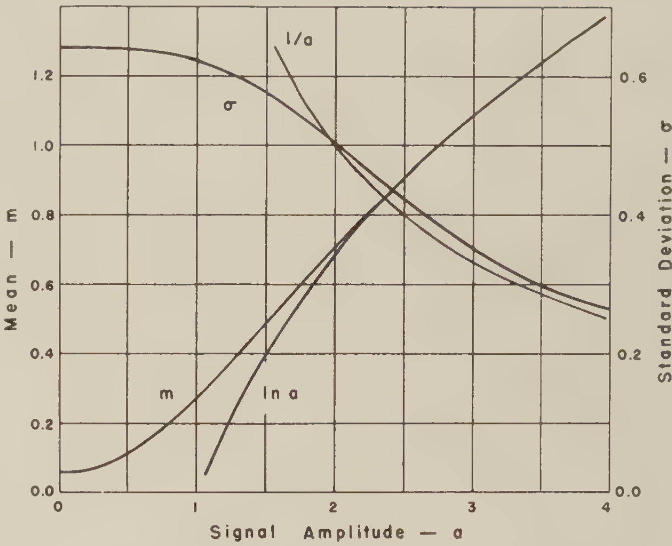

Fig. 1—The mean, m, and standard deviation, σ, for the distribution of the output of a logarithmic detector (to base e) when the input is the sum of a sine wave of amplitude, a, and thermal noise of unit rms value. The asymptotic values, $\ln a$ and $1/a$, respectively, are also shown.

This completes the work except for a minor compli- cation. We saw fit to account for the effect of the antenna pattern, which modulates the sequence of pulses as the antenna sweeps through the target azimuth. We assumed that the pattern was Gaussian in shape and that the k pulses were equally spaced between its "two-way half- voltage" points. Thus, the mean and standard deviation of the sum of the k pulses are sums weighted by the Gaussian pattern.

This makes it a little awkward to compare results for the logarithmic detector with those for the square-law detector, since Marcum assumed a "square" pattern, but it should suffice merely to adjust the value of a in Marcum's results so that the average power received is equal in the two cases.

Discussion

The results are shown in Figs. 2 and 3. The abscissas are the relative range R/R_0, where R_0 is that range for which the signal-to-noise power ratio is unity (with the antenna broadside to the target). Thus, assuming a fourth power range law, we have

$$\frac{R}{R_0} = (a^2/2)^{-1/4} = (0.707a)^{-1/2}.$$

Fig. 2 shows the probability of detection when $P_N = 10^{-5}$ for the two cases $k = 10$ and $k = 100$ pulses integrated. Fig. 3 shows the same when $P_N = 10^{-10}$ and includes Marcum's results for the square-law detector as dashed lines.

We conclude that one suffers some loss of sensitivity by use of a logarithmic detector. The amount of the loss may be stated in terms of the equivalent loss of power; that is, the reduction in power of a conventional

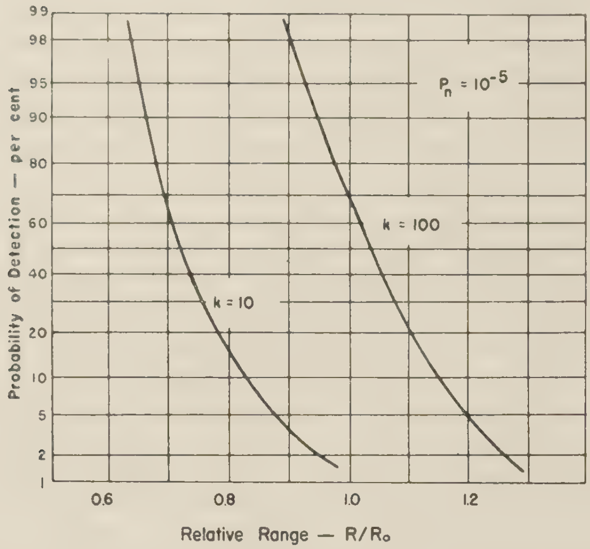

Fig. 2—The probability of detection vs relative range after k pulses are integrated; false alarm probability, 10^{-5}.

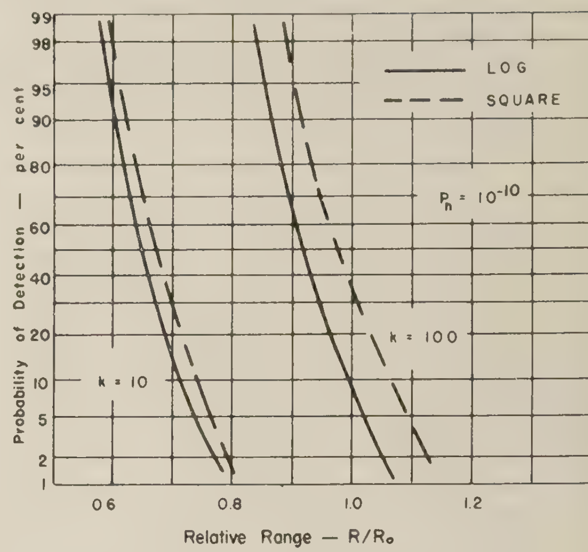

Fig. 3—The probability of detection vs relative range after k pulses are integrated, false alarm probability, 10^{-10}. The solid lines are for the logarithmic detector; the dashed ones, for the square-law detector, (taken from Marcum,[1] and shifted to account for Gaussian antenna pattern).

system which inflicts the same loss of sensitivity. For 10 pulses integrated the loss is about 0.5 db; for 100 pulses integrated, about 1.0 db. We may also say, for 1 pulse the loss is zero. Thus, crudely, it appears that the loss (in db) is proportional to the logarithm of the number of pulses integrated.

It is very likely that in most cases the advantages of a logarithmic detector will far outweigh the power loss.

Acknowledgment

The author wishes to thank Bendix Radio and the Cornell Aeronautical Laboratories for their permission to publish this work. He also wishes to acknowledge the assistance of James Dokas of Bendix Radio with many mathematical details.

Correction

Saburo Muroga, author of "On the Capacity of A Noisy Continuous Channel," which appeared on pages 44–51 of the March, 1957 issue of these Transactions, has requested the editors to make the following corrections to his paper, which were suggested by Prof. Y. Moriwaki of Tokyo University.

On page 47, (39) should be

$$C = -H(n) + \tfrac{1}{2}\log (e\pi/\gamma)$$

and (41) should read

$$\gamma = \tfrac{1}{2}(P + \overline{n^2} - (\bar{n})^2)^{-1}.$$

Eq. (42) should be

$$C = 2W[-H(n) + \tfrac{1}{2}\log_2 (\pi/\gamma)e]$$

and (45) should be changed to

$$C = W\left[\log_2 \frac{2\pi}{e}\left(1 + \frac{P}{N}\right)\right].$$

The term P in (46) and (47) should be replaced by $P + 2N$.

Two lines above (48), the second equation should be $\gamma = -\tfrac{1}{2}(P + N)$. In the first paragraph on the right side of page 48, the footnote references to Muroga's work should be 1 and 3.

On page 51, two lines above (71), the term y' should be x'.

Envelopes and Pre-Envelopes of Real Waveforms*

J. DUGUNDJI†

Summary—Rice's formula[1] for the "envelope" of a given signal is very cumbersome; in any case where the signal is not a single sine wave, the analytical use and explicit calculation of the envelope is practically prohibitive. A different formula for the envelope is given herein which is much simpler and easier to handle analytically. We show precisely that if $\hat{u}(t)$ is the Hilbert transform of $u(t)$, then Rice's envelope of $u(t)$ is the absolute value of the complex-valued function $u(t) + i\,\hat{u}(t)$. The function $u + i\,\hat{u}$ is called the pre-envelope of u and is shown to be involved implicitly in some other usual engineering practices.

The Hilbert transform $\hat{u}$ is then studied; it is shown that $\hat{u}$ has the same power spectrum as u and is uncorrelated with u at the same time instant. Further, the autocorrelation of the pre-envelope of u is twice the pre-envelope of the autocorrelation of u.

By using the pre-envelope, the envelope of the output of a linear filter is easily calculated, and this is used to compute the first probability density for the envelope of the output of an arbitrary linear filter when the input is an arbitrary signal plus Gaussian noise. An application of pre-envelopes to the frequency modulation of an arbitrary waveform by another arbitrary waveform is also given.

Section I

We recall Rice's formulation[2] of the envelope of a multichromatic[3] waveform $u(t)$. He starts by writing $u(t)$ in the form

$$u(t) = \sum_n c_n \cos(\omega_n t + \varphi_n)$$

and then selects a frequency q called the "midband frequency." Using this selected frequency, he writes

$$u(t) = \sum_n c_n \cos[(\omega_n - q)t + \varphi_n + qt]$$

$$= I_c \cos qt - I_s \sin qt$$

where

$$I_c = \sum_n c_n \cos[(\omega_n - q)t + \varphi_n]$$

$$I_s = \sum_n c_n \sin[(\omega_n - q)t + \varphi_n].$$

The expression

$$R(t) = [I_c^2 + I_s^2]^{1/2}$$

is termed by Rice the "envelope of $u(t)$ referred to frequency q."

This formulation has several defects. First, to obtain the envelope, one must expand the given multichromatic waveform in the form above. Second, one must select a "midband frequency" q; it is not immediately evident whether or not a choice $q' \neq q$ leads to the same $R(t)$. Finally, the explicit calculation of $R(t)$ is a formidable task.

* Manuscript received by the PGIT, May 5, 1957. This work was done while the author was consultant to RCA, Los Angeles, Calif.
† University of Southern California, Los Angeles, Calif.
[1] S. O. Rice, *Bell Sys. Tech. J.*, vol. 23, p. 1; 1944.
[2] *Ibid.*, p. 81.
[3] That is a waveform that can be written in the form $\sum_n c_n \cos(\omega_n t + \varphi_n)$, $-\infty < t < +\infty$.

In this paper, we intend to give an equivalent, but more direct formulation of the concept "envelope." This is done by introducing the idea of the pre-envelope of a given real waveform, which is a complex-valued function whose absolute value turns out to be exactly the envelope of the given waveform. Applications of the pre-envelope are given: 1) to show that $R(t)$ depends only on the given waveform and not on the seemingly extraneous concept of "midband frequency,"[4] 2) to give direct methods for obtaining the envelope of the output of a linear filter when the input is an arbitrary waveform, 3) to calculate explicitly the first probability density for the envelope of the output of an arbitrary linear filter when the input is an arbitrary signal plus Gaussian noise, and 4) to give a definition of frequency modulation when both the waveform $u(t)$ and the modulating function $m(t)$ are arbitrary.

The paper is divided into six parts. Section II contains the essential facts about Hilbert transforms that are used; proofs will be found in Titchmarsh.[5] In Section III the definitions of pre-envelope and envelope are given, and the equivalence of our envelope with that of Rice is established. Section IV contains further properties of pre-envelopes and, in particular, shows how one is naturally led to their consideration. In Sections V-VII the applications cited above are given.

We remark that Hilbert transforms have been used in electrical engineering before, notably by Gabor,[6] Woodward,[7] and Ville;[8] the concept "pre-envelope" is in fact identical with the "signal analytique" of Ville, but our use and purpose differ from his.

Section II

Given a real-valued function $u(t)$ on $-\infty < t < +\infty$, its Hilbert transform $\hat{u}(t)$ is defined by

$$\hat{u}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(\xi)}{t - \xi}\, d\xi,$$

where the principal value of the integral is always used. All functions considered here will be assumed to have Hilbert transforms. One readily verifies

A. The Hilbert transform of $\cos(\omega t + \varphi)$ is $\sin(\omega t + \varphi)$.

It is proved in Titchmarsh[5] that, under suitable conditions, $\hat{\hat{u}} = -u$.

[4] This independence of "midband frequency" can of course also be seen directly from Rice's definition without using pre-envelopes.
[5] E. C. Titchmarsh, "Introduction to the Theory of Fourier Integrals," Oxford University Press, New York, N. Y.; 1937.
[6] D. Gabor, *J.IEE*, pt. 3, vol. 93, p. 429; 1946.
[7] P. M. Woodward, "Probability and Information Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1953.
[8] J. A. Ville, "Cables et Transmissions," (2nd ed.) no. 1, p. 16; 1948.

B. The Hilbert transform of a Hilbert transform is the negative of the original function.

C. If $U(f) = \int_{-\infty}^{\infty} u(t) \exp(-2\pi i f t) dt$ is the Fourier transform of $u(t)$, then the Fourier transform $L(f)$ of $\hat{u}(t)$ is

$$L(f) = \begin{cases} -iU(f) & f > 0 \\ 0 & f = 0 \\ +iU(f) & f < 0. \end{cases} \qquad i = \sqrt{-1}$$

D. The convolution

$$w(t) = v(t) * u(t) = \int_{-\infty}^{\infty} v(\xi) u(t - \xi)\, d\xi$$

has Hilbert transform

$$\hat{w}(t) = v(t) * \hat{u}(t).$$

Section III

Definition: Let $u(t)$ be a real waveform. The pre-envelope $z(t)$ of $u(t)$ is the complex-valued function $z(t) = u(t) + i\hat{u}(t)$. The envelope of $u(t)$ is the absolute value $|z(t)|$ of its pre-envelope.

1) This definition of envelope gives the same result as that of Rice, whenever Rice's definition is applicable.

In fact, with the notations of Section I, we start by writing $u(t)$ in the form

$$u(t) = \sum_n c_n \cos(\omega_n t + \varphi_n).$$

Forming the pre-envelope $z(t)$ of $u(t)$ by using A., one gets

$$z(t) = \sum_n c_n \cos(\omega_n t + \varphi_n) + i \sum_n c_n \sin(\omega_n t + \varphi_n).$$

Referring to the frequency q and noting that

$$\sum_n \sin[(\omega_n - q)t + \varphi_n + qt] = I_s \cos qt + I_c \sin qt,$$

one finds

$$z(t) = [I_c \cos qt - I_s \sin qt] + i[I_s \cos qt + I_c \sin qt]$$
$$= [I_c + iI_s] \exp[iqt]$$

which shows that

$$|z(t)| = R(t)$$

and establishes the result.

Since $z(t)$ depends only on $u(t)$, one obtains the following corollary.

Corollary. Rice's envelope is completely independent of the "midband frequency" q that is selected.

Section IV

A motivation for using pre-envelopes is given here. Further, the power spectrum of $\hat{u}$ and the cross correlation of u, $\hat{u}$ are calculated; these will be needed in the sequel.

The motivation for Hilbert transforms comes from 2) and 3) as follows:

2) The Fourier transform of $z = u + i\hat{u}$ is

$$Z(f) = \begin{cases} 2U(f) & f > 0 \\ U(f) & f = 0 \\ 0 & f < 0 \end{cases}$$

where $U(f)$ is the Fourier transform of $u(t)$.

Since $Z(f) = U(f) + iL(f)$, where L is the Fourier transform of $\hat{u}$, the result is immediate from Section II-C.

The important thing is that the inverse of 2) is also true.

3) Let $z(t)$ be any (necessarily complex-valued) function with Fourier transform $Z(f)$ vanishing for all $f < 0$. Then z is the pre-envelope of its real part. That is, if u = real part of z, then $z = u + i\hat{u}$.

To see this, define

$$U(f) = \begin{cases} \frac{1}{2}Z(f) & f > 0 \\ 0 & f = 0 \\ \frac{1}{2}Z^*(-f) & f < 0 \end{cases}$$

where the asterisk denotes (as it will throughout this paper) "complex conjugate." Setting

$$u(t) = \int_{-\infty}^{\infty} U(f) \exp[+2\pi i f t]\, df,$$

one easily shows that $u(t) = u^*(t)$, so that u is real-valued. Form the pre-envelope $\mu = u + i\hat{u}$ of u. By 2), its Fourier transform is precisely $Z(f)$ except possibly at $f = 0$, and this implies at once that $\mu = z$.

Now, it is standard engineering practice in considering frequency spectra to double the positive frequencies and cut the negative ones on the grounds that the positive frequencies are the physically meaningful ones and the (mathematical) negative frequencies merely reflect the positive ones in complex conjugate form. From 2) and 3), it is evident that the mathematical counterpart of this physical practice is to form the pre-envelope of the waveform considered. The fact of 1) is a further reinforcement of the utility of the concept of "pre-envelope."

For any two waveforms u, v, whether real or complex-valued, their cross correlation R_{uv} is defined by

$$R_{uv}(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u^*(\xi) v(t + \xi)\, d\xi$$

and their cross-power spectrum W_{uv} is the Fourier transform of R_{uv}:

$$W_{uv}(f) = \int_{-\infty}^{\infty} R_{uv}(t) \exp[-2\pi i f t]\, dt.$$

It is well known[9] that

E. $$R_{uv}(t) = R_{vu}^*(-t).$$

[9] James, Nichols, and Phillips, "Theory of Servomechanisms," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 25; 1950. See p. 279.

ne autocorrelation R_{uu} of u is written simply R_u.

4) The cross correlation $R_{u\hat{u}}$ is precisely $\hat{R}_u$, the Hilbert ansform of R_u.

In fact, rewriting $\hat{u}(t)$ in the form

$$\hat{u}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(t+x)}{-x}\,dx,$$

ne has, since u is real-valued,

$$R_{u\hat{u}}(t) = \frac{1}{\pi}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} d\xi \int_{-\infty}^{\infty} \frac{u(\xi)u(\xi+x+t)}{-x}\,dx.$$

ssuming interchangeability of the order of integration nd limits gives

$$R_{u\hat{u}}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dx}{-x} \cdot\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} u(\xi)u(x+t+\xi)\,d\xi\right\}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_u(x+t)}{-x}\,dx = \hat{R}_u(t)$$

hich establishes the result.

Corollary. $R_{u\hat{u}}(-t) = -R_{u\hat{u}}(t)$. In particular, u and $\hat{u}$ e completely uncorrelated at the same time instant, ., $R_{u\hat{u}}(0) = 0$.

For, by using E. and observing that R_u is real-valued, ne has

$$R_{u\hat{u}}(-t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_u(x-t)}{-x}\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_u(t-x)}{-x}\,dx;$$

nd changing the variable of integration to $\xi = -x$ ves the desired statement.

Corollary. The cross-power spectrum of u and $\hat{u}$ is

$$W_{u\hat{u}}(f) = \begin{cases} -iW_u(f) & f > 0 \\ 0 & f = 0 \\ +iW_u(f) & f < 0. \end{cases}$$

his is immediate from C. and 4).

We now turn to the study of $R_{\hat{u}}$ and $W_{\hat{u}}$.

5) u and $\hat{u}$ have precisely the same autocorrelation and ower spectrum.

From 4) one has $\hat{R}_u(t) = R_{u\hat{u}}(t)$. Now, starting from and its Hilbert transform $\hat{\hat{u}} = -u$, one finds from 4) ain that

$$\hat{R}_{\hat{u}}(t) = R_{\hat{u}\,\hat{\hat{u}}}(t) = -R_{\hat{u}u}(t).$$

sing E. and the first corollary above shows

$$\hat{R}_{\hat{u}}(t) = -R_{u\hat{u}}(-t) = \hat{R}_u(t)$$

hich is what was asserted.

For the pre-envelope itself, one gets immediately from and 5) the following:

6) Let $z(t) = u(t) + i\hat{u}(t)$. Then

$$R_z(t) = 2[R_u(t) + i\hat{R}_u(t)]$$

$$W_z(f) = \begin{cases} 4W_u(f) & f > 0 \\ 2W_u(f) & f = 0 \\ 0 & f < 0. \end{cases}$$

Section V

Application to filters is now given.

7) Let $u(t)$ be a real input to a linear filter having a real impulsive response function $r(t)$. Then the pre-envelope of the output is obtained by taking the pre-envelope of u as input.

In fact, with the pre-envelope $z = u + i\hat{u}$ as input, one obtains for output the convolution

$$w = r * z = r * u + ir * \hat{u}.$$

But $u_0 = r*u$ is the output when u alone is input, and by D. one has $r*\hat{u} = \hat{u}_0$. Thus, $w = u_0 + i\hat{u}_0$, proving the assertion.

Corollary. In terms of the frequency response function $Y(f)$ of the filter, the pre-envelope of the output when u is taken for input is

$$2\int_0^{\infty} Y(f)U(f)\exp(2\pi ift)\,df$$

where $U(f)$ is the Fourier transform of $u(t)$.

For, by using 7), one need only compute the output when the pre-envelope $z = u + i\hat{u}$ is input. Since the Fourier transform of a filter output equals the Fourier transform of the filter input times frequency response function, the output in the case to hand has Fourier transform $Y(f)Z(f)$. By 2),

$$Y(f)Z(f) = \begin{cases} 2Y(f)U(f) & f > 0 \\ Y(f)U(f) & f = 0 \\ 0 & f < 0. \end{cases}$$

Defining $Y(0)U(0) = 0$, which does not affect the inverse Fourier transform of $Y(f)Z(f)$, one has for output the desired formula.

Using a slightly different viewpoint, this corollary says that the pre-envelope of the output is obtained by taking u alone as input and redefining the filter frequency response by doubling the positive frequencies and killing the negative ones. As another simple application.

8) Let $u(t)$ be a waveform having frequency spectrum in the bands $f_0 - \frac{1}{2}W < |f| < f_0 + \frac{1}{2}W$. Then the square of the envelope is frequency limited to $|f| \le W$.

Indeed, if $z = u + i\hat{u}$ is the pre-envelope of $u(t)$, we are interested in

$$K(f) = \int_{-\infty}^{\infty} |z(t)|^2 \exp[-2\pi ift]\,dt.$$

Now,

$$K(f) = \int_{-\infty}^{\infty} z(t)z^*(t) \exp[-2\pi i f t]\, dt = Z(f) * Z^*(-f)$$

$$= \int_{f_0-\frac{1}{2}W}^{f_0+\frac{1}{2}W} Z(\xi)Z^*(\xi - f)\, d\xi$$

and this is readily seen, by 2), to vanish outside $|f| \leq W$.

We remark that the hypotheses of 8) do not allow one to draw the conclusion that the envelope itself is band-limited; indeed, there seems to be no physical reason that it should be so.

Section VI

The first probability density for the envelope of the output of a linear filter, when the input is arbitrary signal plus Gaussian noise with zero mean, will be computed.

It is well known[10] that if Gaussian noise having zero mean and power spectrum $W(f)$ is passed through a linear filter having frequency response $Y(f)$, then the output 1) is Gaussian with zero mean and 2) has power spectrum $|Y(f)|^2 W(f)$. From this 2) we find at once that the variance of the output distribution, which is the value of the autocorrelation at $t = 0$, is

$$\sigma^2 = \int_{-\infty}^{\infty} |Y(f)|^2 W(f)\, df.$$

The probability density of the output N is therefore

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{N^2}{2\sigma^2}\right].$$

Let $\{n(t)\}$ be sample functions for the noise, and consider the two-dimensional stochastic process Q whose samples are the ordered pairs $[n(t), \hat{n}(t)]$, the second term always being the Hilbert transform of the first. Observe that if $n_0(t)$ is the output of the filter when $n(t)$ is input, then by 7) the output when $\hat{n}(t)$ is applied is the Hilbert transform $\hat{n}_0(t)$ of $n_0(t)$. The two-dimensional distribution of the output $(N, \hat{N})$ when the Q process is put through the filter is now to be determined.

We have seen that N is Gaussian with zero mean and variance σ^2. Now $\hat{N}$ is also Gaussian with zero mean and variance σ^2. That $\hat{N}$ is Gaussian with zero mean is seen by using for the $\{n(t)\}$ the same functions as does Rice[11] and applying A. to each. That the variance of $\hat{N}$ is σ^2 results from 5), since this tells us that the functions $\{\hat{n}(t)\}$ have the same power spectrum as the $\{n(t)\}$.

Putting the sample $[n(t), \hat{n}(t)]$ through the filter gives, as has been remarked, the output $[n_0(t), \hat{n}_0(t)]$, the second variable being the Hilbert transform of the first. According to the corollary in 4) $n_0(t)$ and $\hat{n}_0(t)$ are always uncorrelated at the same instant of time. It follows at once that N and $\hat{N}$ are independent, so that their joint probability density is given by

$$p(N, \hat{N}) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{N^2 + \hat{N}^2}{2\sigma^2}\right]. \qquad (1)$$

With the preliminaries aside, we prove the following:

9) Let signal $u(t)$ plus Gaussian noise with zero mean and power spectrum $W(f)$ be put through a filter having frequency response function $Y(f)$. Then the envelope R of the output has first probability density

$$p(R) = \frac{R}{\sigma^2} \exp\left[-\frac{R^2 + |z(t)|^2}{2\sigma^2}\right] I_0\left[\frac{R \cdot |z(t)|}{\sigma^2}\right]$$

where I_0 = Bessel function of zero order and purely imaginary argument,

$$\sigma^2 = \int_{-\infty}^{\infty} |Y(f)|^2 W(f)\, df,$$

$$|z(t)| = 2\left|\int_0^{\infty} Y(f)U(f) \exp[2\pi i f t]\, df\right|$$

= envelope of output if signal alone is input.

In fact, let u_0 be the output when u alone is input, and n_0 be the output when the sample function $n(t)$ is input; then, by 7), the pre-envelope of the output of signal plus noise has sample functions $(u_0 + n_0) + i(\hat{u}_0 + \hat{n}_0)$, where, according to the corollary of 7) one has $u_0 + i\hat{u}_0 = z$. The envelope of the output is therefore

$$R = [(u_0 + n_0)^2 + (\hat{u}_0 + \hat{n}_0)^2]^{1/2}.$$

In view of the previous discussion, the problem is reduced to calculating the probability density of

$$R = [(u_0 + N)^2 + (u_0 + \hat{N})^2]^{1/2}$$

where N, $\hat{N}$ are distributed as in (1).

The substitution

$$R \cos\theta = u_0 + N$$

$$R \sin\theta = \hat{u}_0 + \hat{N}$$

gives

$$N^2 + \hat{N}^2 = R^2 + u_0^2 + \hat{u}_0^2 - 2R[u_0 \cos\theta + \hat{u}_0 \sin\theta]$$

and the method now used is the same as that of Rice.[12] Observing that

$$u_0^2 + \hat{u}_0^2 = |z|^2$$

one gets from (1)

$$p(R, \theta)\, dR\, d\theta = \frac{R}{2\pi\sigma^2} \exp\left[-\frac{R^2 + |z|^2}{2\sigma^2}\right] \cdot \exp\left[\frac{R}{\sigma^2}(u_0 \cos\theta + \hat{u}_0 \sin\theta)\right] dR\, d\theta.$$

[10] *Ibid.*, p. 288.
[11] Rice, *op. cit.*, p. 48.
[12] *Ibid.*, p. 106.

One now integrates out the θ by setting

$$u_0 = \rho \cos \mu$$

$$\hat{u}_0 = \rho \sin \mu$$

where $\rho = |z|$, verifying that $u_0 \cos \theta + u_0 \sin \theta = \cos(\theta - \mu)$, and then noting that,

$$\int_0^{2\pi} e^{\alpha \cos \theta}\, d\theta = 2\pi I_0(\alpha).$$

The result is established.

Section VII

Application to frequency modulation will now be given.

Definition: Let $u(t)$ be a given real waveform. For any real waveform $m(t)$, we will say that the function

$$\mu(t) = u(t) \cos m(t) - \hat{u}(t) \sin m(t)$$

is $u(t)$ frequency modulated by $m(t)$.

Before giving a justification of this definition, observe that if $u(t) = \cos 2\pi ft$ and $m(t) = m \sin 2\pi\alpha t$, then from A. it follows at once that our definition reduces to the usual definition of frequency modulation:[13] $\mu(t) = \cos(2\pi ft + m \sin 2\pi\alpha t)$.

We also notice that if $z(t)$ is the pre-envelope of $u(t)$, and if $\mathfrak{R}$ denotes "real part of," then

$$\mu(t) = \mathfrak{R}\{z(t) \exp(im(t))\}.$$

However, we must remark that the complex wave $z(t) = \exp[im(t)]$ is *not* the pre-envelope of $\mu(t)$.

One is led to this definition in the following natural way. We have

$$u(t) = \mathfrak{R}\{|z(t)| \exp[i\psi(t)]\} = |z(t)| \cos \psi(t)$$

where

$$\psi(t) = \arctan \frac{\hat{u}(t)}{u(t)}$$

and $|z(t)|$ is the envelope of $u(t)$. Thus, $u(t)$ has been written as a cosine wave with a "slowly" varying amplitude and an instantaneous frequency; frequency modulation in the usual sense[13] says that the frequency modulated wave is

$$\begin{aligned}\mu(t) &= |z(t)| \cos[\psi(t) + m(t)] \\ &= \mathfrak{R}\{|z(t)| \exp[i(\psi(t) + m(t))]\} \\ &= \mathfrak{R}\{|z(t)| \exp[i\psi(t)] \cdot \exp[im(t)]\} \\ &= \mathfrak{R}\{z(t) \exp[im(t)]\}.\end{aligned}$$

This physical reasoning, together with the fact that it coincides with the usual definition whenever both are applicable and because this proposed definition applies in cases where the usual one does not, leads us to propose our definition as a generalization of the usual notion of frequency modulation.

As a simple application, we calculate the spectrum of an arbitrary real waveform $u(t)$ frequency modulated by $b \sin 2\pi\alpha t$. Let $z(t)$ be the pre-envelope of $u(t)$. Observe first that if $\Gamma(f)$ is the Fourier transform of the complex waveform $\gamma(t) = z(t) \exp[ia \sin 2\pi\alpha t]$ then the spectrum of its real part, which is what we are seeking, is $\frac{1}{2}[\Gamma(f) + \Gamma^*(-f)]$; thus it suffices to compute $\Gamma(f)$. Recalling that

$$\exp[ib \sin 2\pi\alpha t] = \sum_{n=-\infty}^{n=\infty} J_n(b) \exp(i2\pi n\alpha t)$$

where J_n, $n \geq 0$ is the ordinary Bessel function of order n and $J_{-n}(x) = (-1)^n J_n(x)$, we have

$$\gamma(t) = \sum_{n=-\infty}^{n=\infty} J_n(b) z(t) \exp[i2\pi n\alpha t].$$

If $Z(f)$ is the Fourier transform of $z(t)$ and $\delta(f)$ is the Dirac delta function, the convolution theorem gives

$$\Gamma(f) = \sum_n J_n(b) Z(f) * \delta(f - n\alpha) = \sum_{n=-\infty}^{\infty} J_n(b) Z(f - n\alpha).$$

To express this directly in terms of $U(f)$, the Fourier transform of the given $u(t)$ according to 2) is:

$$Z(f) = (1 + \operatorname{sgn} f) U(f)$$

where

$$\operatorname{sgn} f = \begin{cases} +1 & f > 0 \\ 0 & f = 0 \\ -1 & f < 0, \end{cases}$$

and we find

$$\Gamma(f) = \sum_{-\infty}^{\infty} J_n(b)[1 + \operatorname{sgn}(f - n\alpha)] U(f - n\alpha).$$

As remarked, $\frac{1}{2}[\Gamma(f) + \Gamma^*(-f)]$ is the required spectrum.

[13] Note the meaning we attach to "frequency modulation;" this is quite often called "phase modulation." In all events we can pass from "phase" to "frequency" modulation easily by using derivatives.

Correspondence

On Weighted PCM and Mean-Square Deviation

In an interesting paper Bedrosian has introduced the concept of weighted pulse-code modulation, wpcm.[1] This differs from normal pcm in that the amplitude of the transmitted pulses representing the binary digits in a pulse-code group are made to depend on the size of the group and on the power of two represented by the individual pulses. In general, the higher the power of two represented by the pulse, the larger the amplitude of the pulse. A comparison between pcm and wpcm system performances, indicating the advantages of wcpm, can be found in Bedrosian's paper.

1) The analysis of a wpcm system involves the determination of the optimal power to be allocated to the transmission of each binary digit, under the assumption of a fixed power per pulse-code group, to minimize the mean-square deviation of the reconstructed signal at the receiver from the signal sample value presented by the information source. If there are N digits per pulse-code group, then use of the Lagrange multiplier method involves the solution of a system of $N + 1$ simultaneous transcendental equations. Bedrosian accomplishes this under the assumption that the average signal power available is sufficiently great to make the occurrence of more than one error per pulse-code group almost zero. This assumption leads to a system of $N + 1$ independent equations for the power allocations and the Lagrange multiplier.

In addition, he assumes that the probability of a digit's being received incorrectly as a function of signal strength, and the *derivative* of this function, are both simply expressible in terms of the error function. This may lead to difficulties in locating the extrema since the smoothed Curve I in Fig. 1 may be a reasonable approximation to Curve II, without the slope of I at a point necessarily being a reasonable approximation to the slope of II at the same point.

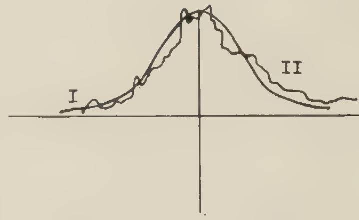

Fig. 1.

2) We wish to present an alternative approach to the analysis, based on the functional equation technique of dynamic programming,[2] in which no assumption of smallness is necessary concerning the probability of error; this is significant for a wpcm system may function satisfactorily with signal power so low that several errors per pulse group occur, the errors being localized to the coefficients of the lower powers of two, which is manifestly not possible with ordinary pcm. In addition, minima are determined without the use of partial derivatives. This permits the probability of error to be given as a function of signal strength either graphically or in tabular form, based upon experimental results, and not necessarily in analytic form. Lastly, the quantization noise is handled in a particularly simple and straightforward fashion.

3) Consider that each pulse-code group consists of N binary digits and that the power P is available for sending the entire group. The probability of the correct reception of a digit if the transmitted power is p is defined to be $g(p)$. Assume that transmission and reception of the individual digits are independent. We wish to determine the power p_i to allocate to the transmission of the coefficient of the ith power of two which results in minimizing the mean-square deviation of the received signal from the signal sample value provided by the source of information, where

$$\sum_{i=0}^{N-1} p_i = P, \qquad (1)$$

$$0 \le p_i, \qquad i = 1, 2, \cdots, N-1. \qquad (2)$$

Assume that the transmitter will transmit a "one" as the coefficient of 2^{N-1} if $a \ge 2^{N-1}$, where a is the signal sample value, and a "zero" if $0 \le a < 2^{N-1}$; call this coefficient c_{N-1}. It transmits a "one" as the coefficient of 2^{N-2} if $a - c_{N-1}2^{N-1} \ge 2^{N-2}$, a "zero" otherwise, and so on.

The reconstructed received signal will be

$$\begin{aligned} s_{N-1} &= 2^{N-1}x_{N-1} \\ &\quad + 2^{N-2}x_{N-2} + \cdots + 2^0 x_0 \\ &= 2^{N-1}x_{N-1} + s_{N-2}, \end{aligned} \qquad (3)$$

where x_k is a random variable assuming only the values zero or one. Since the probability of a correct transmission of the kth digit with power p_k is $g(p_k)$,

$$\text{prob}\,\{x_k = c_k\} = g(p_k). \qquad (4)$$

The mean-square deviation of the received signal from the signal sample value a is

$$\begin{aligned} E\{(a - s_{N-1})^2\} &= E\{(a - 2^{N-1}x_{N-1})^2\} \\ &\quad - 2[a - E\{2^{N-1}x_{N-1}\}]E\{s_{N-2}\} \\ &\quad + E\{s_{N-2}^2\}. \end{aligned} \qquad (5)$$

By completing the square, this becomes

$$\begin{aligned} E\{(a - s_{N-1})^2\} &= E\{(a - 2^{N-1}x_{N-1})^2\} \\ &\quad - [a - E\{2^{N-1}x_{N-1}\}]^2 \\ &\quad + E\{(a - E\{2^{N-1}x_{N-1}\} \\ &\quad - s_{N-2})^2\} \\ &= \text{var}\,\{a - 2^{N-1}x_{N-1}\} \\ &\quad + E\{(a - E\{2^{N-1}x_{N-1}\} \\ &\quad - s_{N-2})^2\}. \end{aligned} \qquad (6)$$

To transform the minimization problem into one involving functional equations, we introduce the function

$f_N(a,P)$ = the expected value of the square deviation of the received signal from the source signal a using an optimal allocation of the power P to the individual digits, each pulse-code group consisting of N digits.

By definition, then,

$$\min_{\{p_i\}} E\{(a - s_{N-1})^2\} = f_N(a, P), \qquad (7)$$

where the minimization is over the set of p_i defined by

$$0 \le p_i, \quad i = 0, 1, \cdots, N-1, \qquad \sum_{i=0}^{N-1} p_i = P. \qquad (8)$$

If we wish to include the effect of a limitation on peak power P_{pk}, we may impose, in addition, the condition that $0 \le p_i \le P_{pk}$. This would be difficult to incorporate in the Lagrange multiplier scheme.

If we now use the principle of optimality[2] and (6), we obtain

$$\begin{aligned} f_N(a, P) &= \\ &\min_{0 \le p_{N-1} \le P} \{\text{var}\,\{a - 2^{N-1}x_{N-1}\} \\ &+ f_{N-1}(a - E\{2^{N-1}x_{N-1}\}, P - p_{N-1})\}, \\ &\qquad N = 2, 3, \cdots. \end{aligned} \qquad (9)$$

If the pulse-code group consists of just one binary digit, all available power is used and we obtain

$$f_1(a, P) = \begin{cases} g(P)a^2 + (1 - g(P))(1 - a)^2, \\ \qquad 0 \le a < 1, \\ g(P)(1 - a)^2 + (1 - g(P))a^2, \\ \qquad 1 \le a < \infty. \end{cases} \qquad (10)$$

Thus the multidimensional optimization problem is converted into a sequence of one-dimensional optimization problems.

[1] E. Bedrosian, "Weighted pcm," this issue, p. 45.
[2] R. Bellman, "Dynamic Programming," Princeton University Press, Princeton, N. J.; 1957.

4) Using (9) and (10) we are able to mpute recursively, using a high-speed gital computer, the sequence of functions two variables $\{f_k(a,P)\}$ and the power locations p_k as functions of k, a, and P. nis computation is straightforward on a gital computer and is simplified by the ct that if $f_N(a,P)$ is to be determined for and P in a certain domain $0 \leq a \leq a_0$, $\leq P \leq P_0$, then all the $f_k(a,P)$ for $k < N$ eed only be determined in subdomains, as seen upon referring to (9).

5) Note that the transmitter is not pro-ded with a feedback link which gives it formation concerning whether or not the evious signal was correctly received, so that the exact state of the system is not known at the transmitter at each stage. Nonetheless, the transmitter is able to transmit optimally with regard to the least mean square deviation criterion by proceeding as if the signal a had been changed purely deterministically to $a - E\{2^{N-1}x_{N-1}\}$.

This observation should find application in other situations involving least mean-square-error criteria. It means that in some situations an equivalent deterministic process can be associated with a stochastic process.

6) The problems involved in optimizing the performance of null-zone reception systems and null-zone reception systems provided with feedback channels[3] can be treated similarly, by use of the functional equation technique. It would be of interest to evaluate the performance of communication systems employing combinations of wpcm, null-zero reception, and feedback channels.

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[3] F. Bloom, S. Chang, B. Harris, A. Hauptschein, and K. Morgan, "Improvement of binary transmission by null-zone reception," Proc. IRE, vol. 45, pp. 963–975; July, 1957.

PGIT News

Continued expansion in our field of terest was demonstrated again by the llowing recent announcement.

An International Association for Cyber-tics has been formed, with the Board of dministration including members from elgium, France, the United Kingdom, and e U. S. A.

A permanent secretariat has been set up ith offices in Namur, Belgium, and the rst International Congress on Cybernetics as held in Namur from June 26–29, 1956. lans for the second congress are being ade now for September, 1958.

WESCON PAPERS DEADLINE SET FOR MAY 1, 1958

Authors wishing to present papers at the 1958 WESCON Convention to be held in Los Angeles, August 19-22, should send 100-word abstracts and either the complete text or a detailed summary, by May 1, to the Technical Program Committee Chairman: Dr. Robert C. Hansen, Microwave Laboratory, Hughes Aircraft Co., Culver City, Calif. There will again be an IRE WESCON Convention Record. Authors will be notified of acceptance or rejection by June 1.

Contributors

Edward Bedrosian (S '51—A '53—M '54 SM '56) was born on May 22, 1922, in hicago, Ill. After receiving the B.S.Ae.E. degree from the Aeronautical University in Chicago in 1942, he worked as an aeronautical engineer at Consolidated-Vultee and Hughes Aircraft Company. During World War II he served in the U. S. Marine Corps as a radar technician.

E. Bedrosian

In 1947, he entered Northwestern University, Evanston, Ill., here he obtained the B.S.E.E. and M.S. degrees in 1949 and 1950, respectively. The study for the M.S. degree was conducted under a fellowship granted by the Airborne Instruments Laboratories. He was awarded the Ph.D. degree by Northwestern in 1953.

He then joined the research staff of the Communications and Electronics Division of Motorola, Inc., Chicago, where he engaged in studies of antennas and communications systems. Later he became a senior staff member at the company's research laboratory in Riverside, Calif., and conducted missile and radar systems analyses. He is currently a member of the Communications Group in the Electronics Department of the Engineering Division of The RAND Corporation, Santa Monica, Calif., participating in countermeasures studies of communications systems.

Dr. Bedrosian is a member of Eta Kappa Nu, Tau Beta Pi, Pi Mu Epsilon, and Sigma Xi.

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Marvin Blum, for a biography and photograph, please see page 208 of the September, 1957, issue of these Transactions.

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J. Dugundji was born in New York, N. Y. on August 30, 1919. He received the B.A. degree from New York University in 1940, and in 1942 began four years of service with the Army Air Force. In 1948, he received the Ph.D. degree in mathe-

matics from Massachusetts Institute of Technology. He was a member of the Institute for Advanced Study from 1951 to 1953, and is presently an associate professor of mathematics at the University of Southern California.

J. Dugundji

Since 1953, he has been a mathematical consultant to the Radio Corporation of America in Los Angeles, Calif.

Dr. Dugundji is a member of Phi Beta Kappa, Sigma Xi, and the American Mathematical Society.

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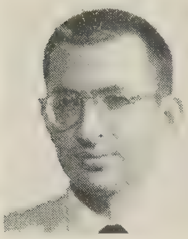
Ben A. Green, Jr. (M '57) was born in Tuscaloosa, Ala. on July 10, 1930. He received the B.S. degree in 1949, and M.S. degree in 1950, from the University of Alabama. From 1950 to 1951, he was an instructor in the Physics Department of Baylor University, Waco, Texas. In 1956, he received the Ph.D. degree from The Johns Hopkins University, Baltimore, Md., for work on positron annihilation in superconductors and other materials.



B. A. Green, Jr.

He was employed by Bendix Radio Corporation from 1956 to 1957, where he participated in design of radar and communication equipment. In 1957, he joined the Metals Research Laboratories of the Electro Metallurgical Company, Division of Union Carbide Corporation, Niagara Falls, N. Y. He is engaged in fundamental research in solid-state physics.

Dr. Green is a member of the American Physical Society and Sigma Xi.

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Robert Lerner (S '48—SM '57) was born in Worcester, Mass., on February 1, 1928. He received the B.S. degree in 1948, and the M.S. degree in 1950, both in electrical engineering from Worcester Polytechnic Institute, Worcester, Mass.

R. Lerner

He then held a Swope Fellowship at the Massachusetts Institute of Technology, Cambridge, Mass., and later became a research assistant in the Research Laboratory of Electronics, working on speech coding. In 1952, he joined the staff of Lincoln Laboratory to work on the application of statistical techniques to the design of radar systems. He is currently concerned with the development of communications systems and with the design of electroacoustic components.

Mr. Lerner is a member of Tau Beta Pi and Sigma Xi.

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Roger Manasse (M '56) was born in New York, N. Y., on April 9, 1930. He received the B.S. degree in physics from the Massachusetts Institute of Technology, Cambridge, Mass., in 1950, and the Ph.D. degree in physics from M.I.T. in 1955.

R. Manasse

During the period 1950–1952, he was a research assistant and then a staff member of the M.I.T. Instrumentation Laboratory where he was engaged in research and development of components for inertial guidance systems.

Since 1954, Dr. Manasse has been a staff member of the M.I.T. Lincoln Laboratory where he has been primarily concerned with the application of noise theory and decision theory to the analysis and design of radar systems.

He is a member of the American Physical Society and Sigma Xi.

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J. A. McFadden was born December 11, 1924, in San Juan, P. R. He received the B.S.E. degree in mathematics in 1945 and the B.S.E. degree in electrical engineering in 1946 both from the University of Michigan, Ann Arbor, Mich. The same university granted him the M.S. and the Ph.D. degrees in physics in 1947 and 1951, respectively.

J. A. McFadden

From 1951 to 1957, he was a physicist at the Naval Ordnance Laboratory, Silver Spring, Md., working primarily in applied mathematics. His specialties have been fluid dynamics, acoustics, and more recently, probability, noise, and stochastic processes.

Since September, 1957, he has been an associate professor of electrical engineering at Purdue University, Lafayette, Ind.

Dr. McFadden is a member of the American Physical Society, Tau Beta Pi, Phi Kappa Phi, and Sigma Xi.

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James Pachares was born on June 14, 1923, in Akron, Ohio. He received the B.S. degree in mathematics from the University of Akron in 1948, and the Ph.D. degree in mathematical statistics from the University of North Carolina, Chapel Hill, N. C., in 1953.

J. Pachares

In August, 1954, Dr. Pachares joined the technical staff at Hughes Aircraft Company, Culver City, Calif., and is presently engaged in numerical and statistical work in the mathematics section of the Systems Analysis Laboratory. In addition, Dr. Pachares is a lecturer in engineering statistics in the extension division of the University of California at Los Angeles.

He is a member of the Institute of Mathematical Statistics, the American Mathematical Society, and RESA.

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Robert Price (S '48—A '54) was born in West Chester, Pa., on July 7, 1929. He received the A.B. degree in physics from Princeton University, Princeton, N. J., in 1950, and the degree of Sc.D. in electrical engineering from the Massachusetts Institute of Technology, Cambridge, Mass., in 1953.

R. Price

At M.I.T. he held an Industrial Fellowship in the Research Laboratory of Electronics and later became a research assistant in the Lincoln Laboratory, where his thesis work on the problems of communicating through multipath disturbances was carried out. Upon completion of this study, he did research in radio-astronomy under a Fulbright award at the Commonwealth Scientific and Industrial Research Organization in Sydney, Australia.

Since 1954, Dr. Price has been a staff member of the Lincoln Laboratory and is currently concerned with various topics in statistical communication theory, decision theory, and noise theory, in addition to communications system design and evaluation.

He is a member of Phi Beta Kappa, Sigma Xi, and the Franklin Institute.

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A. J. F. Siegert, for a photograph and biography, please see page 82 of the March, 1957 issue of these Transactions.

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R. A. Silverman (M '54—SM '58), for a biography and photograph, please see page 209 of the September, 1957 issue of these Transactions.

INFORMATION FOR AUTHORS

Authors are requested to submit editorial correspondence or technical manuscripts to the Publications Chairman for possible publication in the PGIT TRANSACTIONS. Papers submitted should include a statement as to whether the material has been copyrighted, previously published, or accepted for publication elsewhere.

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All technical manuscripts and editorial correspondence should be addressed to Laurin G. Fischer or George A. Deschamps, Federal Telecommunication Labs., 492 River Road, Nutley 10, N. J. Local Chapter activities and announcements, as well as other nontechnical news items, should be addressed to Nathan Marchand, Marchand Electronic Labs., Riversville Road, Greenwich, Conn.